# ON THE BRAIDS FOR $8_{10}$ KNOT 

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SUMMARY: This paper is concerned with $8_{10}$ knots and its braids. The braids structure plays a very important role in Knots Theory. In view of this structure, we obtain braids for that knot, give the representations of Artin and examine Garside Word problem. Then we examine the positivity structure for these knots.

Key words: Braids, Positive word, Representations of Braids.
ÖZET: Bu makale $8_{10}$ düğümü ve onun örgüsü ile ilgilidir. Düğüm teorisinde önemli kavramlardan birisi olan örgü kavramını kullanarak $8_{10}$ düğümünün örgüsünün Artin ve Garside temsilini elde edeceğiz. Daha sonra pozitiflik kavramı irdelenecektir.

Anahtar Kelimeler: Örgü, Pozitif kelime, Örgü Temsili.
INTRODUCTION: The word problem in $\mathrm{B}_{\mathrm{n}}$ was solved by Artin in (Artin, E., 1925). His solution was based on his knowledge of structure of the kernel of the map $\phi$ from from $B_{n}$ to the symetric group $\sum_{\mathrm{n}}$ which sends the generator $\delta_{\mathrm{i}}$ to the transposition ( $i, i+1$ ). The Conjugacy problem in $B_{n}$ was also posed in (Artin, E., 1925), also its importance for the problem of recognizing knots and links algorithmicaly was noted, however it took 43 years before progress was made. In a different, but equally foundational manuscript (Garside, F.A., 1969), he discovered a new solution to the word problem which then led him to a related solution to the conjugacy problem (and also [Birman, J., 1998]).

A somewhat different question is the shortest word problem, to find a representative of the word class which has shortest length in the Artin generators. It was proved in (Paterson, M.S., 1991) that this problem in $\mathrm{B}_{\mathrm{n}}$ is at last as hard as an NP-complete problem. Garside and Thurston and Birman, using new generators, will be able to solve the word problem.

We gave the Braid which is related to Artin and Garside generators and investigated positivity structure for $8_{10}$ knot.

An $n$-braid is a very particular example of an (n)- tangle. On the top and base of a cube, B, mark out n points, $A_{1}, A_{2}, \cdots, A_{n}$ and $A_{1}^{\prime}, A_{2}^{\prime}, \cdots, A_{n}^{\prime}$ respectively.

These points may be arbitrarily placed, however we shall express them in terms of specific coordinates.

Firstly, the coordinates for $B$ in $R^{3}$ are,

$$
B=\{(x, y, z) \mid 0 \leq x, y, z \leq 1\}
$$

Let us choose $A_{1}, A_{2}, \cdots, A_{n}, A_{1}^{\prime}, A_{2}^{\prime}, \cdots, A_{n}^{\prime}$ as follows,

$$
\begin{gathered}
A_{1}=\left(\frac{1}{2}, \frac{1}{n+1}, 1\right), \cdots, A_{n}=\left(\frac{1}{2}, \frac{n}{n+1}, 1\right) \\
A_{1}^{\prime}=\left(\frac{1}{2}, \frac{1}{n+1}, 1\right), \cdots, A_{n}^{\prime}=\left(\frac{1}{2}, \frac{n}{n+1}, 0\right)
\end{gathered}
$$

By the construction each $A_{i}^{\prime}$ is directly below the corresponding $A_{i}$ Figure 1. Now join the $A_{1}, A_{2}, \cdots, A_{n}$, to $A_{1}^{\prime}, A_{2}^{\prime}, \cdots, A_{n}^{\prime}$ bymeans of n curves in B . As usual, they are joined in such a way that these curves (including the end points) do not mutually intersect each other. We will call these polygonal arcs strings.


Figure 1
Suppose that we divide the cube into two parts by an arbitrary plane E that is paralel to base of the cube $B$. Then, if $E$ intersects each string at one and only one point, we say that these $n$ strings in $B$ are an $n$-braid.

Suppose that $B_{n}$ is the set of all $n$-braids (to be more precise all the equivalence class of these braids). For two elements in $B_{n}$, i.e., for two $n$-braids $\alpha$ and $\beta$, it is possible to define a product for two $n$ braids. First, glue the base of the cube that contains $\alpha$ to the top face of the cube that contains $\beta$. The gluing together of the two
cubes produces a rectengular solid in which there exists a braid that has been created from vertical juxtaposition of $\alpha$ and $\beta$. Figure 2.

So far we have described a set $\mathrm{B}_{\mathrm{n}}$ a product in this set, and also that the associativity holds in the set. The unit e is simply the trivial braid and irrespective of the braid $\alpha$, $\alpha e=\alpha$ and similarly, $e \alpha=\alpha$.

In order to find an inverse for an arbitrary $\alpha$, let us consider the mirror image, $\alpha^{\bullet}$ of $\alpha$. Then we may write that $\alpha \alpha^{\bullet}=e$ and $\alpha^{\bullet} \alpha=e$. Therefore we now have all the essentials for $B_{n}$ to be a group.


Figure 2
A fundemental result on the braid group $B_{n}$ is that it has only the following two types of relations called the fundemental relations:

$$
\begin{aligned}
& \text { (1) } \sigma_{i} \sigma_{j}=\sigma_{j} \sigma_{i} \quad(|i-j| \geq 2) \\
& \text { (2) } \sigma_{i} \sigma_{i+1} \sigma_{i}=\sigma_{i+1} \sigma_{i} \sigma_{i+1}(i=1,2, \cdots, n-2)
\end{aligned}
$$

Collecting together the various relations that we have discussed so far, we may write $B_{n}$ in terms of its generators $\sigma_{1}, \sigma_{2}, \cdots, \sigma_{n-1}$ and these fundemantal relations,

$$
B_{n}=\binom{\sigma_{1}, \sigma_{2}, \cdots, \sigma_{n-1}\left|\sigma_{i} \sigma_{j}=\sigma_{i} \sigma_{i}\right| i-j \mid \geq 2}{\sigma_{i} \sigma_{i+1} \sigma_{i}=\sigma_{i+1} \sigma_{i} \sigma_{i+1}(i=1,2, \cdots, n-2)}
$$

where the right hand side is said to be a presentation of $B_{n}$ (Artin generators).

In Garside genarators, i.e., positive braids, i.e., braids which are positive powers of the generators, Garside introduced the fundemantal braid $\Delta$ :

$$
\begin{equation*}
\Delta=\left(\delta_{1} \delta_{2} \cdots \delta_{n-1}\right)\left(\delta_{1} \delta_{2} \cdots \delta_{n-2}\right) \cdots\left(\delta_{1} \delta_{2}\right) \delta_{1} \tag{1}
\end{equation*}
$$

He showed that every element $W \in B_{n}$ can be represented algoritmically by a word W of the form $\Delta^{r} P$, where $r$ is an integer and $P$ is a positive word, and $r$ is a maximal for all such representations (Birman, J., 1998).

Lemma 1: The word $\Delta^{\#}$ in $\mathrm{S}_{\mathrm{n}}$ has the following properties:
(i) For each word $V \in S_{n}, \Delta^{\#} V=\widehat{V} \Delta^{\#}$.
(ii) For each j $(1 \leq j \leq n-1), D\left(\Delta^{\#}\right)$ contains a word with initial letter $\mathrm{s}_{\mathrm{j}}$ and word with final letter $\mathrm{s}_{\mathrm{j}}$ (Birman, J., 1974).

From now on the semigroup $\mathrm{S}_{\mathrm{n}}$ will be identified with its image $e\left(S_{n}\right)$ in $B_{n}$.

Lemma 2: For each $\mathrm{j}\left((1 \leq j \leq n-1)\right.$, there is a positive word $X_{j}$ such that $\sigma_{j}^{-1}=\Delta^{-1} X_{j}$. Also, $\Delta \sigma_{j}^{-1}=\sigma_{n-j}^{-1} \Delta$ (which, together with Lemma 1 , implies that $\Delta^{-1} V=\bar{V} \Delta^{-1}$ for every braid word $V$ ) (Birman, J., 1974).

Theorem 1 (Garside Solution to the Word Problem in $\mathrm{B}_{\mathrm{n}}$ ): If $\beta \in B_{n}$ then $\beta$ is represented by unique word of the form $\Delta^{m} \bar{P}$, where the integers $m$ and the positive word $\bar{P}$ are computed from any representative $\sigma_{\mu_{1}}^{\varepsilon_{1}} \cdots \sigma_{\mu_{r}}^{\varepsilon_{r}}$ of the word $\beta$ in the following manner:
(i) List the positive word $X_{1}, \cdots, X_{n-1}$ whose existence is established by Lemma 1 .
(ii) Replace every letter $\sigma_{\mu_{i}}^{-1}$ which occurs in the braid word $\sigma_{\mu_{1}}^{\varepsilon_{1}} \cdots \sigma_{\mu_{r}}^{\varepsilon_{r}}$ by $\Delta^{-1} X_{\mu_{i}}$.
(iii) Using the property $\Delta^{-1} V=\widehat{V} \Delta^{-1}$ (Lemma 2) collect all $\Delta^{-1}$ 's introduced in (ii) at the left, so that $\beta$ is represented by a word of the form $\Delta^{k} P_{0}$, where $\mathrm{P}_{0}$ is positive. Note that $k \leq 0$.
(iv) Construct $\mathrm{D}\left(\mathrm{P}_{0}\right)$
(v) $\quad \ln \mathrm{D}\left(\mathrm{P}_{0}\right)$, choose a word $\Delta^{h} P$ such that h is maximal. Let $\mathrm{m}=\mathrm{h}+\mathrm{k}$ (Note that $h \geq 0$ )
(vi) Construct $\mathrm{D}(\mathrm{P})$. Let $\bar{P}$ be base of $\mathrm{D}(\mathrm{P})$.

Proof: See (Birman, J., 1974).

Now, let $\alpha$ be a braid and let us connect, by a set of parallel arcs that lie outside the square, the points $A_{1}, A_{2}, \cdots, A_{n}$ on the top of a rectengular diagram of a braid $\alpha$ to the points $A_{1}^{\prime}, A_{2}^{\prime}, \cdots, A_{n}^{\prime}$ respectively, on the bottom of the same diagram. Then in a natural way we form regular diagram of knot or link from a braid. A knot that has been created in this way is said to be a knot, K , created from the braid $\alpha$.

Theorem 2 (Alexander's Theorem): Given arbitrary (oriented) knot (or link), then it is equivalent (with orientation) to a knot (or link) that has been formed from a braid (Murasugi, K., 1996).

Proof: Let $D$ be an oriented regular diagram of a knot K. Firstly cut the $D$ at a point (not a crossing point) $P_{0}$, and then pull the loose ends apart so that we now have a $(1,1)$ tangle. Figure 3.

We shall show that we can change this tangle into a braid $\alpha$. The knot, in a sense induced as described previously from the braid, is equivalent to K . If the tangle T has m local maxima, then it also has m local minima. In the case $\mathrm{m}=0, \mathrm{~T}$ is a 1 braid and so no proof is required.


Figure 3
So suppose that $\mathrm{m}>0$, then there exists an arc ab in T , which we may say "is rising upwards", connecting a local minimum to a local maximum b. Figure 4.


Figure 4
Further we may assume that $a b$ intersects with the other parts of the tangle at $n$ places. Let us now mark $\mathrm{n}+1$ points on ab, i.e., $a=a_{0}, a_{1}, \cdots, a_{n}=b$, such that the arc $a_{i} a_{i+1}$ intersects only one part of the tangle, see Figure 4(b). Next replace the arc $a_{0} a_{1}$ by the much larger arc $a_{0} P_{1}^{\prime} P_{1} a_{1}$. The large arc $P_{1} P_{1}^{\prime}$ lies outside the tangle T , and the $\operatorname{arcs} a_{0} P_{1}^{\prime}$ and $a_{1} P_{1}$ are selected in such a way that if $a_{0} a_{1}$ passes over (or under) the other segment, then they also pass over (or under) all the other segments. The result of the above manipulations is a (2,2) tangle. Figure 4(b).

It follows immediately that the oriented knot obtained by joining (outside the square) the four endpoints of this $(2,2)$ tangle by curves is equivalent to original knot. By using the same methods as above, with regard to the arcs $a_{1} a_{2}, a_{2} a_{3}, \cdots, a_{n-1} a_{n}$ we will eventually form a $(n+1, n+1)$ tangle $T$ that does not have a local minimum and maximum at $a$ and $b$ respectively, and further has at most $m-1$ local maxima anda minima. Continuing this procedure, we shall finally create a tangle that has no local maxima and minima. This tangle is our required braid (Murasugi, K., 1996).

Definition: A knot is called positive, if it has a positive diagram, i.e., diagram with all crossings positive as follows or its braids has no $\sigma_{i}^{-1}$ elements.

$\operatorname{sign}(c)=+1$

$\operatorname{sign}(c)=-1$

Lemma 3: If a positive n -string braid $\beta$ involves a single occurence of some braid generators $\delta_{j}(1 \leq j \leq n-1)$, then it has same closure as some positive ( $n-1$ ) string braid (Birman, J., 1974).

Lemma 4: If the closure of a positive $n$-braid is a knot or non-separated link, then its Conway polynomial is of degree $\mathrm{c}-\mathrm{n}+1$, where c is its number of braid crossing, n braids string (Birman, J., 1974).

Lemma 5: If K is a prime positive knot which has a Conway polynomial of degree 2 m , then a minimal string positive braid p which closes to K has at most $4 m-3$ crossing (Buskirk, J.V., 1983).

Now, we will find braids for $8_{10}$ knots. Regular diagram of $8_{10}$ is given in Figure 5 .


Figure 5
Then Theorem 2 is applied for the knot. Then we got 4-braids for this knot. 4 braid is given in Figure 6.


We get $\beta$-braid for $8_{10}$ knot, i.e.,

$$
\beta=\sigma_{2}^{-1} \sigma_{3}^{2} \sigma_{2} \sigma_{1}^{-2} \sigma_{2}^{2} \sigma_{1}^{-2} \sigma_{2}^{-1} \sigma_{3}^{-1}
$$

At the same time, we will write the Garside generators for this braid. We applied Theorem I to $\beta$ braid which is shown in Figure 6. Then we write:

$$
\beta=\Delta^{-7} A \sigma_{3}^{2} \sigma_{s} B C \sigma_{2}^{2} B C A
$$

where,

$$
A=\sigma_{3} \sigma_{1} \sigma_{2} \sigma_{3} \sigma_{1}, B=\sigma_{3} \sigma_{2} \sigma_{1} \sigma_{3} \sigma_{2}, C=\sigma_{1} \sigma_{2} \sigma_{3} \sigma_{1} \sigma_{2}
$$

These equations are the representation of $8_{10}$ knots (Garside Rep.).
Now we examine the positivity structure. Lemma 4 gives us that $\beta$ braid's Conway polynomial is of degree 7. By Figure $6, \mathrm{c}=12, \mathrm{n}=4$. But its Conway polynomial is of degree 6 . Also, Lemma 5 gives us its braids at most 9 crossings. But its braids has 12 crossings. Then $8_{10}$ knot is not positive.

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