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THE BIJECTIVITY CRITERION, CONTINUUM HYPOTHESIS, AND NUMBER SEQUENCE AND SERIES WITHOUT SOME DOGMAS

КРИТЕРИЙ БИЕКТИВНОСТИ, КОНТИНУУМ ГИПОТЕЗА И ЧИСЛОВЫЕ ПОСЛЕДОВАТЕЛЬНОСТИ И ЧИСЛОВЫЕ РЯДЫ БЕЗ ДОГМ

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Abstract. In Introduction, we give the Alternative decision of David Hilbert's first Problem. Our paper contains demonstrative denying a hypothesis about the existence of a bijection between a set of positive integers and its own subset. This statement is a basis of an alternative methodology, in which a significant tool is the concept of *C–(m, k)–pair of natural variables*. We define *e–divergence* and *w–convergence* of number sequences with this methodology. In particular, the equality $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ is a characteristic feature *for a w–converging number sequence.* We proved that the set of Cauchy sequences coincides with the set *of w–converging* ones and, hence, contains a subset of the infinite large sequences; everyone from them converges to corresponding infinite large number (***ILN***). In particular, a harmonic series converges to the some *ILN*, and the necessary attribute of some number series convergence is also a sufficient one.

Аннотация. Во введении мы даем альтернативное решение первой проблемы Д. Гильберта. Наша статья содержит доказательное отрицание гипотезы о существовании биекции между множеством натуральных чисел и его собственным подмножеством. Это утверждение является основой альтернативной методологии, в которой важным инструментом является понятие *C*–(*m, к*)–пара натуральных переменных, определены *e–расходимость* и *w–сходимость* числовых последовательностей в этой методологии. В частности, равенство $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$ является характеристическим свойством для *w*–сходящейся числовой последовательности. Мы доказали, что множество последовательностей Коши совпадает с множеством *w*–сходящихся последовательностей и, следовательно, содержит подмножество бесконечных больших последовательностей, каждая из которых сходится к соответствующему бесконечно большому числу (***ILN***). В частности, гармонический ряд сходится к некоторому *ILN*, а необходимый признак сходимости каждого числового ряда является также достаточным.

Keywords: bijectivity criterion, continuum hypothesis, C–(m, k)–pair, Cauchy sequences, e–divergence, w–convergence, infinite large number, alternative methodology, infinite larger number, quantity π(x) of prime numbers, maximal prime, alternative number series, dogmas.

Ключевые слова: критерий биективности, континуум–гипотеза, C–(m, k)–пара, последовательности Коши, e–расходимость, w–сходимость, бесконечно большие числа, альтернативная методология, количество π(x) всех простых чисел, наибольшее простое число, знакопеременный числовой ряд, некоторые догмы.

Introduction

Now we shall consider that for arbitrary sets A, B there exists a set $\boldsymbol{F}(A,B) \triangleq \{f | f: A \to \boldsymbol{B}\}$ of all mappings from A into $\boldsymbol{B}$. A mapping $\varphi: A \to B$ is named the surjective one if $\varphi(A) = \boldsymbol{B}$, i.e. $\varphi \in \boldsymbol{Su}(A,B)$. A mapping $f \in \boldsymbol{F}(A,B)$ is said to be injective one if

$$f(a) = f(q) \Rightarrow q = a, \text{ here the } q \text{ is a symbol of variable,} \tag{0.1}$$

i. e. $f \in \boldsymbol{In}(A,B)$. A different yet equivalent definition of the injective mapping named an injection too has the following kind: $a \neq q \Rightarrow f(a) \neq f(q)$. If $f \in \boldsymbol{In}(A,B) \cap \boldsymbol{Su}(A,B)$ then the f is said to be a bijective one, or a bijection, i.e. $f \in \boldsymbol{Bi}(A,B)$. In this case we say the sets A and B are bijective sets and write either $A \sim B$ or $|A| = |B|$. By virtue of these definitions we have for arbitrary sets A, B

$$\boldsymbol{Bi}(A,B) = \boldsymbol{In}(A,B) \cap \boldsymbol{Su}(A,B). \tag{0.2}$$

Theorem 0.1 Let

$$A = \bigcup_{i \in J} A_i, J \subset N, \ A_i \cap A_j = \emptyset \text{ at } i \neq j \tag{0.3}$$

be any partition of the set A into not crossed subsets A_i. Then a mapping $f \in F(A,B)$ $\forall i$ defines a partial mapping $f_i: A_i \to \boldsymbol{B}$**,** and if $\forall i \ f_i \in \boldsymbol{In}(A_i, \boldsymbol{B})$ then $f \in \boldsymbol{In}(A,B)$.

At the first we prove that $f_k(A_k) \triangleq B_k \subset B \Rightarrow B_i \cap B_j = \emptyset$ at $i, j, k \in J$ $i \neq j$ by means verification of an implication (01) and so on.

Theorem 0.2. (The bijectivity criterion). A mapping $\varphi: A \to \boldsymbol{B}$ is a bijection if and only if any partition (0.3) of set A into not crossed subsets A_i holds the following two conditions:

$$1) \ \forall i \ \varphi_i \in \boldsymbol{In}(A_i, B), \ 2) \text{ if } \ C \triangleq \bigcup_{i \in J} B_i, J \subset N, \text{ then } C = B. \tag{0.4}$$

Sufficiency□□ of Proof. Now we must show the implication $(0.4) \Rightarrow \varphi \in (\boldsymbol{In}(A,B) \cap \boldsymbol{Su}(A,B))$. At the first, Theorem 0.1 proves that $\varphi \in \boldsymbol{In}(A,B)$. Now second condition in (0.4) holds $\varphi \in \boldsymbol{Su}(A,B)$.

Necessity of Proof. At present we shall prove that $\varphi \in \boldsymbol{Bi}(A,B)$ holds both 1) and 2) in the condition (0.4). Let $\varphi \in In(A,B)$ and $A = \bigcup_{i \in J} A_i, J \subset N, \ A_i \cap A_j = \emptyset$ at $i \neq j$. Then we have $\varphi(A) = \varphi\left(\bigcup_{i \in J} A_i\right) = \bigcup_{i \in J} \varphi(A_i)$. Let $\varphi_i \triangleq \varphi|_{A_i}: A_i \to B$, so $\varphi_i \in In(A_i, B)$ and we have 1) from (0.4). let $\varphi_i(A_i) \triangleq B_i \subset B$ and $\bigcup_{i \in J} B_i, \triangleq C$. It is obvious that $C \subseteq B$. Now we shall show that an existing of strong inclusion $C \subset B$ contradicts to the mapping φ surjectivity. Let $C^* \triangleq B \backslash C \neq \emptyset$, so $\exists c_0 \in C^* \subset B$. Thus there exists a pair $\{k, a_0: k \in J, a_0 \in A_k\}$ such that $a_0 \triangleq \varphi^{-1}(c_0) \in A_k$. Thus $c_0 = \varphi_k(a_0) \in B_k \subset C$ and, therefore $c_0 \notin C^*$.

One of the first alternative variants of this theorem was published in [1, p. 92].

Theorem 0.2 has following below consequence as

Theorem 0.3 The infinite sets are divided into classes of equivalence as well as the finite sets to within of one element.

Really, let $A \triangleq (B \cup \{h\}, h \notin B)$, $A_1 \triangleq B$, $A_2 \triangleq \{h\}$. Now $\forall f \in \boldsymbol{F}(A,B)$ $f \notin \boldsymbol{In}(A,B)$ because there exists in B a pair $\{b_0, b_1\}: f(b_1) = b_0 = f(h)$.

Theorem 0.3 gives the Alternative decision of David Hilbert's first Problem which is the Dedekind–Cantor's Continuum Hypothesis (CH). We called R. Dedekind as co–author of the CH on the basis of G. Cantor's correspondence with him [2, pp 327–372]. Published correspondence between G. Cantor and R. Dedekind contains XLIX letters, Cantor wrote 35 letters from them.

In the letter II (29.11.1873) Cantor wrote that no matter how he was inclined to think that any one–valued correspondence between a set (n) and a set (x) can not be established, nevertheless he can not to find a reason for this, although it is simple perhaps and namely that is what takes him ...

So now we think our Theorem 03 is the answer to Cantor on his address to Dedekind.

Here we followed to Paul Cohen's forecast about continuum–hypothesis (CH) [3, IV.13]: "A point of view which the author feels may eventually come to be accepted is that CH is obviously false".

Now there is the time and place to say some words about the finite and the infinite in Mathematics without some dogmas too. Namely here we can note the priority of a concept of set ordering before the concept of the finite–infinite, as it was done, for example, in our textbook [4, 3.5]**.**

Definition 0.1 A linearly ordered set is said to be as the finite set, if it is either empty, or a singleton, or each its subset except trivial has two extreme elements: the smallest and the largest. Linearly ordered set we call the infinite one, if at least one its subset has less than two extreme elements.

1. *C*–exact pairs and the mapping $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ surjectivity

To begin with, we introduce a novel concept $C-(m,k)-pair$ of natural variables. Let sets $A \subset \boldsymbol{N}$ and $B \subset \boldsymbol{N}$ be infinite sets with either $A \cap B = \emptyset$ or $\cap B \supseteq \emptyset$ and $E \triangleq A \cup B \subseteq \boldsymbol{N}$. Further, Let $\Psi \triangleq \{(m, k)$: $(m, k) \in (A, B)\} \subset (A, B)\}$ be the set of pairs neighboring in the E elements m and k.

Definition 1.1 The pair (m, k) of natural variables $m \in A$ and $k \in B$ is said to be $C-(m,k)-pair$ if there exists such a number $C \in \boldsymbol{N} \backslash \{1\}$ that the every pair $(m, k) \in \Psi$ holds the inequality

$$|m-k| < C. \tag{1}$$

Condition (1) has the following equivalent form of record:

$$\exists \widetilde{C},\ \tilde{C} \geq C, (\forall k \in B\ \exists m \in A): k = m + p(m), p(m) \notin \boldsymbol{Z}, |p(m)| < \tilde{C}. \tag{2}$$

Let as above the $\boldsymbol{In}(\boldsymbol{N},\boldsymbol{N})$ *be a set of injective functions* $\varphi: \boldsymbol{N} \to \boldsymbol{N}$. *In this item we will consider the functions* $\varphi \in \mathbf{In}(\mathbf{N}, \mathbf{N})$ *on default. A sequence* $\xi \triangleq (1, n_1, n_2, \ldots, n_i, \ldots)$ *of natural numbers* n_i *is said to be a sequence with a limited step if there exists such number* $C_\xi \in \boldsymbol{N}$ *that* $\forall i \in \boldsymbol{N}(\xi)$, *where* $\boldsymbol{N}(\xi) \triangleq \{i: \exists n_i \in \xi\} \subseteq \boldsymbol{N}$, $0 < n_i - n_{i-1} < C_\xi$, $n_0 \triangleq 1$. *Further, let a set* N_i *be defined as* $\{1, 2, \ldots, n_i\}$. *The sequence* ξ *and a mapping* $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ *define two number sequences*

$$\delta_i \triangleq max_{n \leq n_i}\{\varphi(n) - n_i\} \geq 0 \quad and \quad d_i \stackrel{\Delta}{=} |D_i| \geq 0, \qquad D_i \triangleq N_i \backslash \varphi(N_i). \tag{3}$$

It is obvious that $|D_i| = |N_i \backslash \varphi(N_i)|$ *and then* $d_i \leq \delta_i$. *Really,* $d_i = \delta_i$ *if and only if* $\forall p, n_i < p < \delta_i + n_i, \exists n \leq n_i: p = \varphi(n)$. *In all other case we have the inequality* $d_i < \delta_i$. *The mapping* $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ *defines a sequence* $\{\varphi_n\}_{n=1}^{\infty}$ *of integers* $\varphi_n \triangleq \varphi(n) - n$ *as well too. If for some sequence* ξ *there exist both* $\delta_\varphi \triangleq sup_{n \in N}(\varphi(n) - n)$ *and* $\delta_\xi \triangleq sup_{i \in N(\xi)}(\delta_i)$ *then we have the obvious inequality*

$$\delta_\xi \leq \delta_\varphi. \tag{4}$$

Now we formulate the direct and obvious corollary of both the definition of set D_i *in (3) and the mapping* $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ *surjectivity as follows:*

Statement 1.1 The necessary condition of the mapping $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ surjectivity has the following two equivalent forms:

$$\forall i \in \boldsymbol{N}(\xi)\ \exists j \in N: D_i \cap D_{i+j} = \emptyset \ and \quad N_i \subset N_{i+j}. \tag{5}$$

Below, for short we say “for almost all *i*” instead of the phrase “except for a final set of indexes “*i”* and we write by definition $\widetilde{\forall} i$. Now we describe the attributes of the surjectivity and antisurjectivity of mapping $\varphi: \boldsymbol{N} \to \boldsymbol{N}$.

Statement 1.2 Sufficient conditions of the surjectivity (*a*) and antisurjectivity (*b*) of the mapping $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ have, accordingly, the following forms

$$(a) \widetilde{\forall} i \in N(\xi)\ d_i = 0, \ (b)\ \forall C \exists i(C) \in N(\xi): d_{i(C)} > C. \tag{6}$$

Proof Each number d_i determines a quantity of such elements n each of which belongs to a subset N_i and does not have a prototype $\varphi^{-1}(n)$ on N_i. Therefore, an unboundedness of sequences $\{d_i\}$ in (*b*) of (6) contradicts to the condition $\varphi(\boldsymbol{N}) = \boldsymbol{N}$ of a mapping φ surjectivity. The condition (*a*) in (6) guarantees the existence of such number i_0 that for the mapping φ the following circuit of implications is valid:

$$\forall j > i_0\ d_j = 0 \Rightarrow D_j = \emptyset \Rightarrow \ \varphi(\boldsymbol{N_i}) = \boldsymbol{N_i} \Rightarrow \varphi(\boldsymbol{N}) = \boldsymbol{N}.$$

We shall speak about an antisurjective injective mapping $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ that it is *potentially impracticable on all set N*. As the examples show, the conditions (6) are not necessary for the surjectivity and antisurjectivity, accordingly, of the function φ. In view of conditions (3)–(6) everyone can prove following below statements easily.

Statement 1.3 The sequences $\{\delta_i\}$ and $\{d_i\}$, $i \in \boldsymbol{N}(\xi)$, defined by means of the pair (ξ, φ), satisfy one and only one of the following three conditions:

$$\text{(a)} \qquad \tilde{\forall} i \in \mathbf{N}(\xi){:}\ (\delta_i = 0) \Leftrightarrow (d_i = 0),$$

$$\text{(b)} \quad (\exists C_1, C_2,\ C_1 \geq C_2 \in \boldsymbol{N}){:}\ (\tilde{\forall} i \in \mathbf{N}(\xi)\ (0 < \delta_i < C_1) \Leftrightarrow (0 < d_i < C_2)), \tag{7}$$

$$\text{(c)} \qquad i \in \mathbf{N}(\xi)\ \ (d_i \to \infty) \Leftrightarrow (\delta_i \to \infty).$$

Statement 1.4 For any injective mapping $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ there exists a sequence ξ of such kind that

$$\delta_\xi = \delta_\varphi. \tag{8}$$

The corollary of Statements 1–4 will be written below.

Theorem 1.1 The boundedness of a sequence $\{\varphi_n\}$ is a necessary condition of the injective mapping $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ surjectivity, i. e. $\varphi(\boldsymbol{N}) = \boldsymbol{N}$ holds

$$\lim_{n\to\infty}(\varphi(n)/n) = 1. \tag{9}$$

Theorem 1.2 The injective mapping: $\varphi^*: \boldsymbol{N} \to \boldsymbol{N}\ \varphi^*(k) \triangleq m_k$, which defines some sequence

$$M^* \triangleq \varphi^*(\boldsymbol{N}) = (m_1, m_2, \dots, m_{k\dots})$$

with an unlimited step $s_k \triangleq m_{k+1} - m_k$, is the antisurjective function or, in other words, *it will be impracticable on all set N*.

Proof Let M^* be sequence with an unlimited step then we have the following condition:

$$\forall C > 0 \exists k(C) \in \boldsymbol{N}(\xi){:}\ |m_{k(C)+1} - m_{k(C)}| > C. \tag{10}$$

Let now $\xi^* = N$ so $\boldsymbol{N}(\xi^*) = \boldsymbol{N}$, and by virtue of (3) we have $n_k = k + 1$ hence $\delta_k^* = \varphi^*(n_k) - n_k$.

Further $\delta_{k+1}^* - \delta_k^* = m_{k+1} - m_k = (m_{k+1} - (k+1)) - (m_k - k) = m_{k+1} - m_k + 1$. Now with (10) we get following inequality for all $k(C) \in \boldsymbol{N}$: $\delta_{k(C)+1}^* - \delta_{k(C)}^* + 1 > C$. Therefore we have

$$\delta_{k(C)+1}^* > C + \delta_{k(C)}^* - 1. \tag{11}$$

The inequality (11) proves an unboundedness of the sequence $\{\delta_k^*\}$ defined by means of this pair (N, φ^*), which follows from the last inequality by virtue of arbitrariness of number C in (10). Therefore, the mapping $\varphi^*: \boldsymbol{N} \to \boldsymbol{N}$, which defines the sequence M^* in this theorem, is an antisurjective one by virtue of (6), (7), (10) and (11).

Theorem 1.2 implies the following statement.

Theorem 1.3 Let $A \triangleq \{k\} \subseteq \boldsymbol{N}$ and $B \triangleq \{m\} \subseteq \boldsymbol{N}$ be infinite subsets of set N. Then there exists such number $C > 0$ that the pair (k, m) of natural variables $k \in A$ and $m \in B$ is C–(m, k)–pair (1).

As the examples show, the necessary conditions (5) and (9) of a surjectivity of an injection $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ are independent ones, hence, any of these conditions cannot be sufficient. However, the following statement below is valid.

Theorem 1.4 The joint realization of conditions (5) and (9) is a sufficient attribute of an injection $\varphi: \boldsymbol{N} \to \boldsymbol{N}$ surjectivity.

2. The convergence of number sequences

A number sequence $(a) \triangleq \{a_n\}_{n=1}^{\infty} \triangleq (a_1, a_2, \dots, a_n, \dots)$ is said to be a fundamental one, or Cauchy sequence ***(CS)*** if

$$(\forall \varepsilon > 0\ \exists\ n(\varepsilon) \in N) : (\forall n, m \geq n(\varepsilon))\ |a_n - a_m| < \varepsilon. \tag{12}$$

The condition (12) is equivalent to the following limit equality:

$$\lim_{n\to\infty}(a_n - a_m) = 0. \tag{13}$$

The condition (13) has (see [5, p. 355]) a more concrete form of record

$$\lim_{\min(m,n)\to\infty}(a_n - a_m) = 0. \tag{14}$$

Corollary of Theorem 1.3 *The pair* (*m*, *n*) *of variables m and n on the conditions* (12)–(14), *each of which defines* Cauchy sequence, *is C*–(*m, k*)*–pair*.

The number sequence (***a***) is said to be converging to a finite number *A*, if $lim(a_n) = 0$. Otherwise, i. e. if $lim(a_n)$ does not exist or it is equal $(\mp\infty)$, the sequence (***a***) is said to be in the traditional analysis divergent one (***DS***). It is obvious: $\{(\boldsymbol{a})\} = \{\boldsymbol{CS}\} \cup \{\boldsymbol{DS}\}$.

As well, how it is accepted in the classical analysis, there is

$$\{\boldsymbol{CS}\}\cap\{\boldsymbol{DS}\}=\varnothing. \tag{15}$$

We introduce a following novel concept for a refutation of equality (15). Let A, B and $\Psi \subset (A, B)$ be as above in item 1.

Definition 2.1 The number sequence (***a***) is said to be *e–divergent one* (*e*–***DS***) if there are such two infinite subsequences $A \subset \boldsymbol{N}$ and $B \subset \boldsymbol{N}$ with $A \cap B = \emptyset$ and $\exists\ (\delta > 0, n^* \in N) : \forall (m, k) \in \Psi,\ m > n^*$, holds the inequality

$$|a_n - a_m \geq \delta. \tag{16}$$

The direct comparison both of conditions (12)–(14) and (16) gives

Theorem 2.1 Any number sequence is either Cauchy sequence, or an *e*–divergent one:

$$\forall(\boldsymbol{a})\ (\boldsymbol{a})\in\{\boldsymbol{CS}\}\cup\{e\text{–}\boldsymbol{DS}\} \text{ and } \{\boldsymbol{CS}\}\cap\{e\text{–}\boldsymbol{DS}\}=\varnothing. \tag{17}$$

It is easy to show, that

$$\{e\text{–}\boldsymbol{DS}\}\subseteq\{\boldsymbol{DS}\}. \tag{18}$$

The example of the sequence $(\boldsymbol{a}) \triangleq \{n^\alpha,\ 0 < \alpha < 1\}_{n=1}^{\infty}$ confirms the following strict inclusion:

$$\{e\text{–}\boldsymbol{DS}\}\subset\{\boldsymbol{DS}\}. \tag{19}$$

Proof The Sequence (***a***) is divergent one, as $0 < \alpha < 1$ holds $\lim_{n\to\infty} n^\alpha = \infty$. On the other hand, by virtue of the Theorem (1.3) the pair (m, k) is any $C-(n, m)-$pair (1) and

$$\exists\ (C > 0,\ q(k) \in Z,\ |q(k)| < C) : m = k + q(k).$$

Now we examine the function $f : R_+ \to R_+$, which is determined by the formula: $f(x) = (x + q(x))^\alpha - x^\alpha$. The value $f(k) = (k + q(k))^\alpha - k^\alpha$ of the function *f* at *x*=*k* coincides with a difference $(m^\alpha - k^\alpha)$ at $m = q + q(k)$. It is easy to show, that $x \to \infty$ holds $\lim_{x\to\infty} f(x) = 0$. Hence, the inequality (1) will be violated, at least, for any one pair $(m_0, k_0) \in (A, B)$, $m_0 > n^*, k_0 > n^*$. Therefore, the sequence $(n^\alpha)\notin\{e\text{–}\boldsymbol{DS}\}$ at $0<\alpha<1$.

Therefore, the strict inclusion (19) takes place instead of condition (18). And, hence, in view of (19), we have the following inequality instead of (15)

$$\{\boldsymbol{CS}\}\cap\{\boldsymbol{DS}\}\neq\varnothing. \tag{20}$$

Now we introduce a concept which has fundamental importance in our theory.

Definition 2.2 The number sequence (**a**) is said to be *w–convergent* (*w*–***CS***) if this sequence (***a***) satisfies following condition

either $\forall \varepsilon > 0\ \exists n(\varepsilon) \in N : \forall n \geq n(\varepsilon)\ |a_{n+1} - a_n| < \varepsilon$, or $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$. (21)

Our textbook [4, 7.1] contains full proofs following theorems.

Theorem 2.2 Any Cauchy sequence (***a***) is *w*–convergent one, *i. e.* {***CS***}⊆{*w*–***CS***}.

Theorem 2.3 Any *w*–convergent sequence is the Cauchy one, *i. e.* {*w*–***CS***}⊆{***CS***}.

Theorems 2.2 and 2.3 compile the following statement:

Theorems 2.4 The set of Cauchy sequences coincides with the set of *w*–convergent sequences:

$$\{\boldsymbol{CS}\} = \{\boldsymbol{w–CS}\}.$$

Theorems 2.4 *follows directly from* both Theorems 2.1 and Definition 2.1 and Definition 2.2 since those definitions holds $\{w–\boldsymbol{CS}\}\}\cap\{e–\boldsymbol{DS}\}=\varnothing$.

Corollary of Theorem 2.4. There exist Cauchy sequences which do not limited by the some finite number.

The study of a sequence $(\boldsymbol{a})\triangleq\{\ln n + C_e + \gamma_n\}$ of the harmonious series sums (see [6, it. 388]) satisfies to condition (21), but its limiting value is more than any finite number. The corollary of Theorem 2.4 motivates an introduction of the following concept.

Definition 2.3 *The limit value of Cauchy sequence* (***a***), *which is not limited by any finite number, is said to be an infinitely large number* **(ILN)**, *defined by this sequence* (***a***).

Let the symbol **Ω** be denoted the set of all ***ILN*.** In the non–standard analysis the ***ILN*** are named (see [7, Ch. 2.1]) as either non–standard, or impracticable, or actually infinite large, or inaccessible numbers.

Proposition 2.1 The sequence $(\boldsymbol{a}) \triangleq \{a_n : a_n = n^{1-\alpha}, \alpha > 0\} \in \boldsymbol{CS}\}$.

Proof $a_{n+1} - a_n = (n+1)^{1-\alpha} - (n)^{1-\alpha} =$

$$= (n+1)/(n+1)^{\alpha} - n/n^{\alpha} < (n+1)/n^{\alpha} - n/n^{\alpha} = 1/n^{\alpha} \to 0.$$

Theorem 2.5 An unlimited differentiated in $\pm\infty$ function $f\colon \boldsymbol{R} \to \boldsymbol{R}$ converges to corresponding ***ILN*** $\Omega(f)$ if and only if $f'(\infty) = 0$.

Proof The passage to limit in mean value theorem which has been written down for function *f*:

$$f(n+1) - f(n) = f'(t)((n+1) - n),\ \ n<t<n+1$$

makes up the proof of Theorem 2.5.

Now we shall receive an important on the Theory of numbers result by means of Theorem 2.5.

The quantity $\pi(x)$ of the prime numbers $p, p < x$, is defined as well know [8, 1.1.5] by the asymptotic formula $\pi(x) = x/lnx + \ o(x/lnx)$

We proved that there exists some ***ILN*** $\triangleq \Omega_{\pi}$ which defines the quantity of all prime numbers:

$$\Omega_{\pi} \triangleq \lim_{x\to\infty}\pi(x), \text{ because } \lim_{x\to\infty}\big(\pi(x)\big)' = 0.$$

Hence we can say that there exists the corresponding ***ILN*** $\triangleq \Omega(\pi)$ for an estimate of maximal prime number.

3. The convergence of alternative number series

Let's designate by the symbol Σ_n the sum of *n* the first members a_i of the number sequence $(\boldsymbol{a})\triangleq(a_n)\triangleq(a_1, a_2, \ldots, a_n, \ldots)$: $\Sigma_n \triangleq a_1 + a_2 + \cdots + a_n$, and the symbol S_n denotes the value of the sum Σ_n. Thus

$$\Sigma_{n+1} = \Sigma_n + a_{n+1}\ ,\ S_{n+1} = S_n + a_{n+1}\,,\ n\in\boldsymbol{N}. \tag{22}$$

Definition 3.1 *The pair of sequences* (Σ_n) *and* (S_n), *defined by* means of equations (22), is said to be *a number series defined by sequence* **(*a*),** *and we shall write*

$$\Sigma_\infty(\boldsymbol{a}) \overset{\Delta}{=} \sum_{n=1}^{\infty} a_n \overset{\Delta}{=} \sum a_n \triangleq a_1 + a_2 + \cdots + a_n + \cdots \overset{\Delta}{=} (\boldsymbol{A}). \tag{23}$$

Here and below the summation at symbol Σ is supposed formally from 1 up to ∞, that means an unlimited opportunity of transition from the Σ_n to the Σ_{n+1}.

Definition 3.2 *The number series* (***A***) *is said to be convergent to the number A, if the number sequence* (s_n) *of the values* s_n *of partial sums* Σ_n *converges to this number A. In this case number A is said to be the sum of series* (***A***), *and we write* $\lim s_n = A$.

The equalities (23) can be written easily in the following way:

$$(\boldsymbol{A}) = \sum a_n = (a_1 + a_2 + \cdots + a_n) + (\sum_{n+1}^{\infty} a_i) \overset{\Delta}{=} \Sigma_n + \rho_n. \tag{24}$$

The value of the infinite sum $\rho_n \overset{\Delta}{=} \sum_{k=n+1}^{\infty} a_k$ in (21), which is called *n–th* rest of series (***A***), shall be denoted by a symbol r_n.

Statement 3.1 The necessary feature of some number series convergence, i.e. $\lim a_n = 0$, is also a sufficient one.

Really, $\lim a_n = \lim(S_n - S_{n-1}) = 0$ is a characteristic criterion (21) of *w*–convergence of a number sequence (S_n), therefore $r_n \to 0$. A reverse implication $(r_n \to 0) \Rightarrow (a_n \to 0)$ is obvious.

The number series (***A***)=$\sum_{i=1}^{\infty} a_i$ is said to be an alternative one, if its quantities of both positive and negative addends are not limited.

Theorem 3.1 The number series (***B***), being any permutation of alternative series (***A***) which converging to some number *A* not absolutely, converges to the same number *A*.

Proof Let a convergent to number *B* number series $B = \sum b_j \overset{\Delta}{=} \tilde{\Sigma}_n + \tilde{\rho}_n$ be resulted by means of a mapping φ: $\boldsymbol{N} \to \boldsymbol{N}$, φ(*k*)=*j*, where $a_k \overset{\Delta}{=} b_j$ from the series (***A***):

$$A = \Sigma_n + \rho_n = \sum_1^n a_j + \sum_{n+1}^{k(n)} a_i + \sum_{k(n)+1}^{\infty} a_i \triangleq \Sigma_n + \sigma(n) + \rho_{k(n)}, \tag{25}$$

where the $k(n)$ *denotes a max*{k: $a_k \triangleq b_j$, j≤n}. *Step by step we shall carry out the mapping φ:* $\boldsymbol{N} \to \boldsymbol{N}$ *and simultaneously build both the sequence* ($\tilde{\Sigma}_n$) *of the partial sums* $\tilde{\Sigma}_n$ *of series* (**B**) *and the sequence* ($\tilde{S}_n$) *of these sums values* $\tilde{S}_n$. *We shall receive the following bellow equality on* n–th *step from the identity* $\sum a_i \equiv \sum a_i$ *in view of (23):*

$$\sum a_i = \Sigma_{k(n)} + \rho_{k(n)} \equiv \Sigma_n + \sigma(n) + \rho_{k(n)} = \tilde{\Sigma}_n + \tilde{\sigma}(n) + \rho_{k(n)}, \tag{26}$$

where the sum $\tilde{\sigma}(n) \overset{\Delta}{=} \sum_{i=n+1}^{k(n)} a_{n_i}$ with $n_i < k(n)$ contains those terms of the partial sum $\Sigma_{k(n)}$ of series (***A***), which don't belong to the partial sum $\tilde{\Sigma}_n$ of series (***B*** (*n*)), and $\sigma(n) = \sum_{i=n+1}^{k(n)} a_i$. Thus with (26), we have $\forall n \in \boldsymbol{N}$ the following equalities:

$$\rho_n - \sigma(n) = \tilde{\rho}_n - \tilde{\sigma}(n), \quad \Sigma_n + \sigma(n) = \tilde{\Sigma}_n + \tilde{\sigma}(n). \tag{27}$$

If we denote by $\tilde{s}(n)$ and $s(n)$ in (26) respectively the values of the sums $\tilde{\sigma}(n)$ and $\sigma(n)$, then we will obtain the number equalities equivalent of (25):

$$s_n + s(n) = \tilde{S}_n + \tilde{s}(n), \quad r_n - s(n) = \tilde{r}_n - \tilde{s}(n). \tag{28}$$

Since $\lim a_n = 0$, $\lim s(n) = 0$, $\lim r_{k(n)} = 0$ at $n \to \infty$ follows from the convergence of series (***A***), then we have $\lim \tilde{r}_n = \lim \tilde{s}(n)$ from the second equality in (28). Now from the first equality in (28) we receive the following result: $\lim s_n = \lim \tilde{S}_n + \lim \tilde{s}(n)$, *i. e.*, $\lim \tilde{r}_n = A - B$ at $n \to \infty$. Thus, in view of $\tilde{S}_n \to B$, $s_n \to A$, we have the required implication: ($\tilde{r}_n \to 0$, $r_n \to 0) \Rightarrow (B = A)$.

In the general case, at $s_n \to$A *and* $r_n \to 0$ *the equivalence* ($\tilde{S}_n \to$B) $\Leftrightarrow$ ($\tilde{r}_n \to$(A–B)) *follows from equality (28), thus we* have

Theorem 3.2 If the sequence (Σ_n^*) of the sum Σ_n^* was constructed arbitrarily from the members of convergent to number *A* alternative series (***A***) and the sequence (S_n^*) of the sums Σ_n^* values S_n^* converges to number *B*, then the sequence (r_n^*) of the values of respective rests ρ_n^* converges to number *A–B* (compare [9, pp 232–233]).

Some results of this paper can be found in the text–book [10], which was published without the consent of the authors, it is readily available, but contains many publishing typos and inaccuracies.

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