Some Embedding Theorems on the Nikolskii-Morrey Type Spaces

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Abstract In the paper the Nikolskii-Morrey type spaces $H^l_{p,\varphi,\beta}(G)$ were introduced and studied. Some embedding theorems are obtained in $H^l_{p,\varphi,\beta}(G)$ with the help of Nikolskii type integral representation.

Keywords: Nikolskii-Morrey type spaces, integral representation, embedding theorems, generalized Holder condition

1 Introduction

In the paper, we introduce a Nikolskii-Morrey type space with parameters. By $H^l_{p,\varphi,\beta}(G)$ we denote the spaces of all functions $f \in L_1^{loc}(G)$ $(m_i > l_i - k_i > 0, i = 1, 2, ..., n)$ with the finite norm

$$\|f\|_{H^l_{p,\varphi,\beta}(G)} = \|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^{n} \sup_{0<h<h_0} \frac{\left\| \Delta_i^{m_i}\left(\varphi_i(h), G_{\varphi(h)}\right) D_i^{k_i} f \right\|_{p,\varphi,\beta}}{\varphi_i(h)^{l_i-k_i}}, \tag{1.1}$$

where

$$\|f\|_{p,\varphi,\beta;G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{x\in G, t>0} \left(\left|\varphi([t]_1)\right|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right), \tag{1.2}$$

$l \in (0,\infty)^n$, $m_i \in \mathbb{N}$, $k_i \in \mathbb{N}_0$, $p \in [1,\infty)$, $[t]_1 = \min\{1,t\}$, $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))$, $\left|\varphi([t]_1)\right|^{-\beta} = \prod_{j=1}^{n} (\varphi_j([t]_1))^{-\beta_j}$, $\beta_j \in [0,1]$, $j = 1, 2, ..., n$.

Denote by $\mathbb{A}$ the set of vector functions $\varphi(t) = (\varphi_1(t), ..., \varphi_n(t))$ with Lebesgue measurable functions $\varphi_j(t) > 0$, $t > 0$, $\lim_{t\to+0} \varphi_j(t) = 0$, $\lim_{t\to+\infty} \varphi_j(t) = \infty$, $j = 1, 2, ..., n$.

For any $x \in R^n$,

$$G_{\varphi(t)}(x) = G \cap I_{\varphi(t)}(x) = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t), \quad j = 1, 2, ..., n \right\}.$$

Let for any $t > 0$, $|\varphi([t]_1)| \le C$, where C is positive constant. Then the embeddings $L_{p,\varphi,\beta}(G) \to L_p(G)$ and $H^l_{p,\varphi,\beta}(G) \to H^l_p(G)$ hold, i.e

$$\|f\|_{p,G} \le c\|f\|_{p,\varphi,\beta;G},$$
$$\|f\|_{H^l_p(G)} \le c\,\|f\|_{H^l_{p,\varphi,\beta}(G)}.$$

Note that the spaces $L_{p,\varphi,\beta}(G)$ and $H^l_{p,\varphi,\beta}(G)$ are Banach spaces. The completeness of these spaces automatically implies from completeness of L_p and H^l_p. The space $H^l_{p,\varphi,\beta}(G)$, when $\varphi_j(t) = t^{\chi_j}$, $\beta_j = \frac{a_j}{p}$ $(j = 1, ..., n)$ coincides with the space $H^l_{p,a,\chi}(G) \equiv H^l_{p,\lambda}$ introduced by J.Ross [13], in the case $\beta_j = 0\,(j = 1, ..., n)$ it coincides with the Nikolski space $H^l_p(G)$.The space $W^l_{p,\varphi,\beta}(G)$ was introduced and studied in [12]. The spaces of such type with different norms were introduced and studied in [2]-[11].

Note some properties of the spaces $L_{p,\varphi,\beta}(G)$.

1. The space $L_{p,\varphi,\beta}(G)$ is complete.

Proof. Let $\{f_i\}_{i=1}^{\infty}$ be the fundamental sequences in $L_{p,\varphi,\beta}(G)$, i.e. for any $i,j\to\infty$

$$\|f_i - f_j\|_{L_{p,\varphi,\beta}(G)} \to 0,$$

It means that $\forall\varepsilon>0$, $\exists n_0\in N$, $\forall i,j>n_0$

$$\|f_i - f_j\|_{L_{p,\varphi,\beta}(G)} < \varepsilon,$$

In other words $\forall\varepsilon>0$, $\exists n_0\in N$, $\forall i,j>n_0$

$$\sup_{x\in G, t>0}\left(|\varphi([t]_1)|^{-\beta}\|f_i - f_j\|_{p,G_{\varphi(t)}(x)}\right) < \varepsilon$$

and for any $x\in G$, $\forall t>0$

$$\|f_i - f_j\|_{p,G_{\varphi(t)}(x)} < \varepsilon$$

i.e. $\{f_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $L_p\left(G_{\varphi(t)}(x)\right)$. The space $L_p(G)$ is complete, therefore there is a function $f_0\in L_p(G), i\to\infty$, for $x\in G$, for any $t>0$, $\forall\varepsilon>0$

$$\|f_i - f_0\|_{L_p(G_{\varphi(t)}(x))} \to \varepsilon$$

then

$$|\varphi([t]_1)|^{-\beta}\|f_i - f_0\|_{L_p(G_{\varphi(t)}(x))} \to \varepsilon$$

i.e.

$$\|f_i - f_0\|_{p,\varphi,\beta;G} < \varepsilon,$$

$$\|f_0\|_{p,\varphi,\beta;G} = \|f_i - f_0\|_{L_{p,\varphi,\beta}(G)} + \|f_i\|_{L_{p,\varphi,\beta}(G)} < \varepsilon_1 + M = \varepsilon_2$$

$\|f_0\|_{p,\varphi,\beta;G} < \varepsilon_2$,$f_0\in L_{p,\varphi,\beta}(G)$.

2. Let G be a bounded domain and $p\le q$; $\varphi(t)\le\psi(t)(t>0)$; $\exists c>0, \forall t\in(0,1), |\psi(t)|^{\beta_1}\le c|\varphi(t)|^{\beta}$, and then $L_{q,\psi,\beta_1}(G)\to L_{p,\varphi,\beta}(G)$ and there exists $C>0$ such that

$$\|f\|_{p,\varphi,\beta;G} \le C\|f\|_{q,\psi,\beta_1;G}.$$

Proof. . For any $t>0$, $x\in G$ we have

$$|\varphi([t]_1)|^{-\beta}\|f\|_{p,G_{\varphi(t)}(x)}$$

$$\le |\varphi([t]_1)|^{-\beta}(mesG_{\varphi(t)}(x))^{\frac{1}{p}-\frac{1}{q}}|\psi([t]_1)|^{\beta_1}\,|\psi([t]_1)|^{-\beta_1}\|f\|_{q,G_{\psi(t)}(x)}$$

and

$$\|f\|_{p,\varphi,\beta;G} \le C\|f\|_{q,\psi,\beta_1;G}.$$

Definition 1.1. The open set $G\subset R^n$ is said to be an open set with condition of flexible φ-horn if for some $\theta\in(0,1]^n$, $T\in(0,\infty)$ for any $x\in G$ there exists the vector-function

$$\rho\left(\varphi(t),x\right) = \left(\rho_1\left(\varphi_1(t),x\right),...,\rho_n\left(\varphi_n(t),x\right)\right),\quad 0\le t\le T$$

with the following properties:

1) For all $j=1,2,...,n$, $\rho_j\left(\varphi_j(t),x\right)$ are absolutely continuous on $[0,T]$, $|\rho_j\left(\varphi_j(t),x\right)|\le 1$ for almost all $t\in[0,T]$,

2) $\rho_j\left(0,x\right)=0,\ x+V(x,\theta) = x+\bigcup\limits_{0\le t\le T}\left[\rho\left(\varphi(t),x\right)+\varphi\left(t\right)\theta I\right]\subset G.$

In particular, for $\varphi(t)=t^{\lambda}$, $\left(t^{\lambda}=\left(t^{\lambda_1},t^{\lambda_2},...,t^{\lambda_n}\right)\right)$ and $\theta_j=\theta^{\lambda_j}$ $(j=1,\ldots,n)$ the set $x+V(x,\theta)$ is called the flexible λ-horn introduced in [1].

Assuming that $\varphi_j(t)\ \ (j=1,2,...,n)$ are also differentiable on $[0,T]$, we can show that for $f\in H_p^l(G)$ determined in n- dimensional domains, satisfying the condition of flexible φ-horn, it holds the following integral representation $(\forall x\in U\subset G)$

$$D^{\nu}f(x)=\overline{f}_{\varphi(t)}^{(\nu)}(x)+(-1)^{|\nu|}\sum_{i=1}^{n}\int_{0}^{T}\int_{R^n}\int_{-\infty}^{+\infty}K_i^{(\nu)}\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t,x))}{\varphi(t)}\right)$$
$$\times\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t,x))}{\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi(t),x)\right)\Delta_i^{m_i}(\varphi_i(\delta)u)$$
$$\times f(x+y+ue_i)\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dtdudy, \tag{1.3}$$

$$\overline{f}_{\varphi(t)}^{(\nu)}(x)=\prod_{j=1}^{n}\varphi_j^{-2-\nu_j}(t)\int_{R^n}\int_{R^n}\Omega^{(\nu)}\left(\frac{y}{\varphi(T)},\frac{\rho(\varphi(t),x)}{2\varphi(t)}\right)$$
$$\times\Omega\left(\frac{z}{\varphi(T)},\frac{\rho(\varphi(t),x)}{2\varphi(t)}\right)f(x+y+z)\,dydz. \tag{1.4}$$

Let $M_i(\cdot,y)\in C_0^{\infty}(R^n)$ be such that

$$S(M_i)\subset I_{\varphi(t)}=\left\{y:|y_j|<\frac{1}{2}\varphi_j(t),\quad j=1,2,...,n\right\}.$$

Assume $0<T\le 1$ is fixed and

$$V=\bigcup_{0<t\le T}\left\{y:\frac{y}{\varphi(t)}\in S(M_i)\right\}.$$

It is clear that $V\subset I_{\varphi(t)}$. Let $U+V\subset G$.

Lemma 1.2. *Let* $1\le p\le q\le r\le\infty;\ \ 0<\eta,\ t<T\le 1,\ \nu=(\nu_1,\nu_2,...,\nu_n),\ \nu_j\ge 0$ *are integers,* $j=1,2,...,n;\ \Delta_i^{m_i}(h)\in L_{p,\varphi,\beta}(G)$ *and let*

$$Q_T^i=\int_0^T\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}\frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}}dt<\infty,$$

$$A(x)=\prod_{j=1}^{n}\int_{R^n}\int_{R^n}f(x+y+z)\Omega^{\nu}\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t),x)}{2\varphi(t)}\right)$$
$$\times\Omega\left(\frac{z}{\varphi(T)},\frac{\rho(\varphi(t),x)}{2\varphi(t)}\right)f(x+y+z)\,dydz, \tag{1.5}$$

$$H_\eta^i(x)=\int_0^\eta L_i(x,t)\prod_{j=1}^{n}(\varphi_j(t))^{\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt, \tag{1.6}$$

$$H_{\eta T}^i(x)=\int_\eta^T L_i(x,t)\prod_{j=1}^{n}(\varphi_j(t))^{\nu_j-2}\frac{\varphi_i'(t)}{\varphi_i(t)}dt, \tag{1.7}$$

where

$$L_i(x,t)=\int_{R^n}\int_{-\infty}^{+\infty}M_i\left(\frac{y}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)}\right)$$
$$\times\zeta_i\left(\frac{u}{\varphi_i(t)},\frac{\rho_i(\varphi_i(t),x)}{2\varphi_i(t)},\frac{1}{2}\rho_i'(\varphi_i(t),x)\right)\Delta_i^{m_i}(\varphi_i(\delta)u)f(x+y+ue_i)\,dudy \tag{1.8}$$

Then for any $\overline{x}\in U$ *the following inequalities*

$$\sup_{\overline{x}\in U}\left\|H_{\eta}^{i}\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_{1}\left\|\left(\varphi_{i}(t)\right)^{-l_{i}}\Delta_{i}^{m_{i}}\left(\varphi_{i}(t),G_{\varphi(t)}\right)f\right\|_{p,\varphi,\beta;G} \times\left|Q_{\eta}^{i}\right|\prod_{j=1}^{n}\left(\psi_{j}\left([\xi]_{1}\right)\right)^{\beta_{j}\frac{p}{q}}, \tag{1.9}$$

$$\sup_{\overline{x}\in U}\left\|H_{\eta T}^{i}\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq C_{2}\left\|\left(\varphi_{i}(t)\right)^{-l_{i}}\Delta_{i}^{m_{i}}\left(\varphi_{i}(t),G_{\varphi(t)}\right)f\right\|_{p,\varphi,\beta;G} \times\left|Q_{\eta T}^{i}\right|\prod_{j=1}^{n}\left(\psi_{j}\left([\xi]_{1}\right)\right)^{\beta_{j}\frac{p}{q}}, \tag{1.10}$$

$$\sup_{\overline{x}\in U}\|A\|_{qU_{\psi(\xi)}(\overline{x})}\leq\|f\|_{p,\varphi,\beta;G}\prod_{j=1}^{n}\left(\varphi_{j}(t)\right)^{-\nu_{j}-(1-\beta_{j}p)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j=1}^{n}\left(\psi_{j}[\xi]_{1}\right)^{\beta_{j}\frac{p}{q}}, \tag{1.11}$$

is hold, where $U_{\psi(\xi)}\left(\overline{x}\right)=\left\{x:\left|x_{j}-\overline{x}_{j}\right|<\frac{1}{2}\psi_{j}\left(\xi\right),j=1,2,...,n\right\}$ *and* $\psi\in N$, C_1, C_2 *are the constants independent of* φ, ξ, η *and* T.

Proof. Applying sequentially the Minkowski generalized inequality for any $\overline{x}\in U$

$$\left\|H_{\eta}^{i}\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq\int_{0}^{\eta}\left\|L_{i}(\cdot,t)\right\|_{qU_{\psi(\xi)}(\overline{x})}\prod_{j=1}^{n}\left(\varphi_{j}(t)\right)^{\nu_{j}-2}\frac{\varphi_{i}'\left(t\right)}{\varphi_{i}(t)}dt, \tag{1.12}$$

and from the Hölder inequality ($q\leq r$) we have

$$\left\|L_{i}(\cdot,t)\right\|_{qU_{\psi(\xi)}(\overline{x})}\leq\left\|L_{i}(\cdot,t)\right\|_{rU_{\psi(\xi)}(\overline{x})}\prod_{j=1}^{n}\left(\psi_{j}(\xi)\right)^{\frac{1}{q}-\frac{1}{r}}. \tag{1.13}$$

Now estimate the norm $\left\|L_{i}(\cdot,t)\right\|_{qU_{\psi(\xi)}(\overline{x})}$. Let X be a characteristic function of the set $S\left(M_{i}\right)=supp\,M_{i}$. Noting that $1\leq p\leq r\leq\infty$, $s\leq r$, represent the integrand function (1.8) in the form

$$\left|\int_{-\infty}^{+\infty}M_{i}\zeta_{i}\Delta_{i}^{m_{i}}f du\right|=\left(\left|\int_{-\infty}^{+\infty}\zeta_{i}\Delta_{i}^{m_{i}}f du\right|^{p}\left|M_{i}\right|^{s}\right)^{\frac{1}{r}} \times\left(\left|\int_{-\infty}^{+\infty}\zeta_{i}\Delta_{i}^{m_{i}}f du\right|^{p}X\left(\frac{y}{\varphi(t)}\right)\right)^{\frac{1}{q}-\frac{1}{r}}\left(\left|M_{i}\right|^{s}\right)^{\frac{1}{s}-\frac{1}{r}}$$

and apply to $|L_i|$ the Hölder inequality $\left(\frac{1}{p}+\left(\frac{1}{p}-\frac{1}{r}\right)+\left(\frac{1}{s}-\frac{1}{r}\right)=1\right)$, we obtain

$$\left\|L_{i}(\cdot,t)\right\|_{r,U_{\psi(\xi)}(\overline{x})}\leq\sup_{x\in U_{\psi(\xi)}(\overline{x})}\left(\int_{R^{n}}\left|\int_{-\infty}^{+\infty}\zeta_{i}\left(\frac{u}{\varphi_{i}(t)},\frac{\rho_{i}\left(\varphi_{i}(t),x\right)}{\varphi_{i}(t)},\frac{1}{2}\rho_{i}'\left(\varphi(t),x\right)\right)\right.\right. \left.\left.\times\Delta_{i}^{m_{i}}\left(\varphi_{i}(t)\right)f\left(x+y+ue_{i}\right)du\right|^{p}\chi\left(\frac{y}{\varphi(t)}\right)dy\right)^{\frac{1}{p}-\frac{1}{r}} \times\sup_{y\in V}\left(\int_{U_{\psi(\xi)}(\overline{x})}\left|\int_{-\infty}^{+\infty}\zeta_{i}\left(\frac{u}{\varphi_{i}(t)},\frac{\rho_{i}\left(\varphi_{i}(t),x\right)}{\varphi_{i}(t)},\frac{1}{2}\rho_{i}'\left(\varphi(t),x\right)\right)\right.\right.$$

$$\times \Delta_i^{m_i} \left(\varphi_i(t)u\right) f\left(x+y+ue_i\right) du|^p \, dx)^{\frac{1}{p}}$$
$$\times \left(\int\limits_{R^n} \left| M_i \left(\frac{y}{\varphi(t)}, \frac{\rho\left(\varphi(t),x\right)}{\varphi(t)}, \rho'\left(\varphi(t),x\right) \right) \right|^s dy \right)^{\frac{1}{s}} . \tag{1.14}$$

For any $x \in U$ we have

$$\int\limits_{R^n} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\left(\varphi_i(t),x\right)}{\varphi_i(t)}, \frac{1}{2}\rho_i'\left(\varphi(t),x\right) \right) \right.$$
$$\times \Delta_i^{m_i} \left(\varphi_i\left(\delta\right)u\right) f\left(x+y+ue_i\right) du|^p \, \chi\left(\frac{y}{\varphi(t)}\right) dy$$
$$\leq \int\limits_{(U+V)_{\varphi(t)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\left(\varphi_i(t),x\right)}{\varphi_i(t)}, \frac{1}{2}\rho_i'\left(\varphi(t),x\right) \right) \Delta_i^{m_i} \left(\varphi_i\left(\delta\right)u\right) f\left(y+ue_i\right) du \right|^p$$
$$\leq \left(\varphi_i(t)\right)^{pl_i} \left\| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\left(\varphi_i(t),x\right)}{\varphi_i(t)}, \frac{1}{2}\rho_i'\left(\varphi(t),x\right) \right) \varphi_i(t)^{-l_i} \right.$$
$$\times \Delta_i^{m_i} \left(\varphi_i\left(\delta\right)u, G_{\varphi(t)}\right) f du\|_{p,G_{\varphi(t)}(x)}^p$$
$$\leq \varphi_i(t)^{p+pl_i} \left\| \varphi_i(t)^{-l_i} \Delta_i^{m_i} \left(\varphi_i\left(\delta\right)u, G_{\varphi(t)}\right) \right\|_{p,\varphi,\beta}^p \prod_{j=1}^n \left(\varphi_j(t)\right)^{\beta_j p} . \tag{1.15}$$

For $y \in V$ and $U_\psi + V \subset G_\varphi$ $(\varphi([t]_1) \leq \psi([t]_1))$

$$\int\limits_{U_{\psi(\xi)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\left(\varphi_i(t),x\right)}{\varphi_i(t)}, \frac{1}{2}\rho_i'\left(\varphi(t),x\right) \right) \Delta_i^{m_i} \left(\varphi_i\left(\delta\right)u\right) f\left(x+y+ue_i\right) du \right|^p dx$$
$$\leq \int\limits_{G_{\varphi(\xi)}(\overline{x})} \left| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\left(\varphi_i(t),x\right)}{\varphi_i(t)}, \frac{1}{2}\rho_i'\left(\varphi(t),x\right) \right) \Delta_i^{m_i} \left(\varphi_i\left(\delta\right)u\right) f\left(x+ue_i\right) du \right|^p dx$$
$$\leq \left(\varphi_i(t)\right)^{pl_i} \left\| \int\limits_{-\infty}^{+\infty} \zeta_i \left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\left(\varphi_i(t),x\right)}{\varphi_i(t)}, \frac{1}{2}\rho_i'\left(\varphi(t),x\right) \right) \right.$$
$$\times \varphi_i(t)^{-l_i} \Delta_i^{m_i} \left(\varphi_i\left(\delta\right)u, G_{\varphi(t)}\right) f du\|_{p,G_{\varphi(t)}(\overline{x})}^p$$

$$\leq \varphi_i(t)^{p+pl_i} \left\| \varphi_i(t)^{-l_i} \Delta_i^{m_i} \left(\varphi_i\left(\delta\right), G_{\varphi(t)}\right) \right\|_{p,\varphi,\beta}^p \prod_{j=1}^n \left(\psi_j\left([\xi]_1\right)\right)^{\beta_j p} \tag{1.16}$$

$$\int\limits_{R^n} \left| M_i \left(\frac{y}{\varphi(t)}, \frac{\rho\left(\varphi(t),x\right)}{\varphi\left(t\right)}, \rho'\left(\varphi(t),x\right) \right) \right|^s dy = \left\| M_i \right\|_s^s \cdot \prod_{j=1} \varphi_j(t). \tag{1.17}$$

From inequalities (1.14)-(1.17) it follows that

$$\left\| L_i(\cdot,t) \right\|_{rU_{\psi(\xi)}(\overline{x})} \leq \left\| M_i \right\|_s \cdot \left\| \left(\varphi_i(t)\right)^{-l_i} \Delta_i^{m_i} \left(\varphi_i\left(\delta\right)\right) f \right\|_{p,\varphi,\beta;G} \left(\varphi_i(t)\right)^{1+l_i}$$
$$\times \prod_{j=1}^n \left(\varphi_j\left(t\right)\right)^{\frac{1}{s}+\beta_j p\left(\frac{1}{p}-\frac{1}{r}\right)} \cdot \prod_{j=1}^n \left(\psi_j\left([\xi]_1\right)\right)^{\frac{\beta_j p}{r}} . \tag{1.18}$$

Inequality (1.11) is proved analogously. □

Inequalities (1.12), (1.13) and (1.18) for $r = q$ and for any $\overline{x} \in U$ reduce to the estimation

$$\left\|H_\eta^i\right\|_{rU_{\psi(\xi)}(\overline{x})} \le C_1 \left\|(\varphi_i(t))^{-l_i} \Delta_i^{m_i} (\varphi_i(\delta)) f\right\|_{p,\varphi,\beta;G} \times \left|Q_\eta^i\right| \prod_{j=1}^n (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}} \quad \left(Q_\eta^i < \infty\right). \tag{1.19}$$

In the case $Q_{\eta,T}^i < \infty$ inequality (1.10) can be proved in the same way.

From inequality (1.18) for $r = q$ and (1.19) we get the inequality ($\forall \overline{x} \in U$)

$$\sup_{\overline{x}\in U} \|L_i\|_{qU_{\psi(\xi)}(\overline{x})} \le C_2 \left\|(\varphi_i(t))^{-l_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G} \cdot \prod_{j=1}^n (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}},$$

$$\sup_{\overline{x}\in U} \left\|H_\eta^i\right\|_{qU_{\psi(\xi)}(\overline{x})} \le C_3 \left\|(\varphi_i(t))^{-l_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G} \cdot \prod_{j=1}^n (\psi_j([\xi]_1))^{\beta_j \frac{p}{q}}.$$

From last inequalities it follows that

$$\|L_i\|_{q,\psi,\beta^1;U} \le C_1' \left\|(\varphi_i(t))^{-l_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G}, \tag{1.20}$$

$$\left\|H_\eta^i\right\|_{q,\psi,\beta^1;U} \le C_2' \left\|(\varphi_i(t))^{-l_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G}. \tag{1.21}$$

C_1' and C_2' are the constants independent of φ.

2 Main Results

Prove two theorems on the properties of the functions from the space $H_{p,\varphi,\beta}^l (G, \lambda)$.

Theorem 2.1. *Let $G \subset R^n$ satisfy the condition of flexible φ-horn, $1 \le p \le q \le \infty$, $\nu = (\nu_1, \nu_2, .., \nu_n)$, $\nu_j \ge 0$ be entire $j = 1, 2, ..., n$, $Q_T^i < \infty$ $(i = 1, 2, ..., n)$ and let $f \in H_{p,\varphi,\beta}^l(G, \lambda)$. Then the following embeddings hold*

$$D^\nu : H_{p,\varphi,\beta}^l(G) \to L_{q,\psi,\beta^1}(G),$$

more precisely, for $f \in H_{p,\varphi,\beta}^l (G, \lambda)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are valid

$$\|D^\nu f\|_{q,G} \le C_1 \left(B(t)\|f\|_{q,\psi,\beta;G} + \sum_{i=1}^n \left|Q_T^i\right| \sup_{0<t<t_0} \left\| \frac{\Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f}{(\varphi_i(t))^{l_i}} \right\|_{p,\varphi,\beta;G} \right), \tag{2.1}$$

$$\|D^\nu f\|_{q,\psi,\beta^1;G} \le C_2 \|f\|_{H_{p,\varphi,\beta}^l(G,\lambda)}, \; p \le q < \infty. \tag{2.2}$$

In particular, if

$$Q_{T,0}^i = \int_0^T \prod_{j=1}^n (\varphi_j(t))^{-\nu_j - (1-\beta_j p)\frac{1}{p}} \frac{\varphi_i'(t)}{(\varphi_i(t))^{1-l_i}} dt < \infty, (i = 1, 2, \ldots, n), \tag{2.3}$$

then $D^{\nu} f(x)$ is continuous on G, i.e

$$\sup_{x\in G}|D^{\nu}f(x)| \le C_1\left(B(t)\|f\|_{p,\varphi,\beta;G} + \sum_{i=1}^{n}\left|Q^i_{T,0}\right| \sup_{0<t<t_0}\left\|\frac{\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f}{(\varphi_i(t))^{l_i}}\right\|_{p,\varphi,\beta;G}\right) \tag{2.4}$$

$0<T\le\min\{1,T_0\}$*, T_0 is a fixed number; C_1, C_2, C_3, C_4 are the constants independent of f, also C_1 and C_3 are independent from T.*

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^{\nu}f$ on G. Indeed, from the condition $Q^i_T<\infty$ for all $(i=1,2,...,n)$ it follows that for $f\in H^l_{p,\varphi,\beta}(G)\to H^l_p(G)$, there exists $D^{\nu}f\in L_p(G)$ and for integral representation (1.3) and (1.4) with the same kernels is valid.

Applying the Minkowski inequality, from identities (1.3) and (1.4) we get

$$\|D^{\nu}f\|_{q,G}\le\left\|f^{(\nu)}_{\varphi(T)}\right\|_{q,G}+\sum_{i=1}^{n}\left\|H^i_T\right\|_{q,G}. \tag{2.5}$$

By means of inequality (1.11) for $U=G$, $M_i=K^i_i$, $t=T$ we get

$$\begin{aligned}\left\|f^{(\nu)}_{\varphi(t)}\right\|_{q,G} &\le \|f\|_{p,\varphi,\beta;G}\prod_{j=1}^{n}(\varphi_j(t))^{-\nu_j-(1-\beta_j p)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j=1}^{n}(\psi_j([\xi]_1))^{\beta_j\frac{p}{q}}\\ &\le C_1A(t)\|f\|_{p,\varphi,\beta;G},\end{aligned} \tag{2.6}$$

and by means of inequality (1.9) for $\eta=T$, $M_i=K^i_i$, $U=G$, we get

$$\left\|H^i_T\right\|_{q,G}\le C_2Q^i_T\left\|(\varphi_i(t))^{-l_i}\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f\right\|_{p,\varphi,\beta;G}. \tag{2.7}$$

Substituting (2.7) and (2.6) in (2.5), we get inequality (1.21). By means of inequalities (1.20) and (1.21) for $\eta=T$ we get inequality (2.2).

Now let conditions (2.3) be satisfied, then take into account identities (1.3), (1.4), from inequality (2.5) we get

$$\left\|D^{\nu}f-f^{(\nu)}_{\varphi(T)}\right\|_{\infty,G}\le C\sum_{i=1}^{n}\left|Q^i_T\right|\sup_{0<t<t_0}\left\|\frac{\Delta_i^{m_i}\left(\varphi_i(t),G_{\varphi(t)}\right)f}{(\varphi_i(t))^{l_i}}\right\|_{p,\varphi,\beta;G}.$$

As $T\to 0$, the left side of this inequality tends to zero, since $f^{(\nu)}_{\varphi(T)}(x)$ is continuous on G and the convergence on $L_{\infty}(G)$ coincides with the uniform convergence. Then the limit function $D^{\nu}f$ is continuous on G. □

Let γ be an n-dimensional vector.

Theorem 2.2. *Let all the conditions of Theorem 1 be fulfilled. Then for $Q^i_T<\infty$ $(i=1,2,...,n)$ the derivative $D^{\nu}f$ satisfies on G the Hölder generalized condition, i.e the following inequality is valid:*

$$\|\Delta(\gamma,G)D^{\nu}f\|_{q,G}\le C\|f\|_{H^l_{p,\varphi,\beta}(G)}\cdot|h(|\gamma|,\varphi;T)|, \tag{2.8}$$

where C is a constant independent of f, $|\gamma|$ and T.

In particular, if $Q^i_{T,0}<\infty$, $(i=1,2,\ldots,n)$, then

$$\sup_{x\in G}|\Delta(\gamma,G)D^{\nu}f(x)|\le C\|f\|_{H^l_{p,\varphi,\beta}(G_{\varphi})}\cdot|h_0(|\gamma|,\varphi,T)|. \tag{2.9}$$

where $h(|\gamma|,\varphi,T)=\max_i\left\{|\gamma|,Q^i_{|\gamma|},Q^i_{|\gamma|,T}\right\}$ $\left(h_0(|\gamma|,\varphi,T)=\max_i\left\{|\gamma|,Q^i_{|\gamma|,0},Q^i_{|\gamma|,T,0}\right\}\right)$

Proof. According to Lemma 8.6 from [1] there exists a domain

$$G_\omega \subset G\left(\omega = \zeta r\left(x\right), \, \zeta > 0 \; r\left(x\right) = \rho\left(x, \partial G\right), \; x \in G\right)$$

and assume that $|\gamma| < \omega$, then for any $x \in G_\omega$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identities (1.3), (1.4) with the same kernels are valid. After the same transformations, from (1.3) and (1.4) we get

$$\begin{aligned}
&\left|\varDelta\left(\gamma, G\right) D^\nu f\left(x\right)\right| \le \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-1-\nu_j} \\
&\times \int_{R^n}\int_{R^n} \left|f\left(x+y+z\right)\right| \left|\varOmega^{(\nu)}\left(\frac{y-\gamma}{\varphi(t)}, \frac{\rho\left(\varphi(t), x\right)}{2\varphi(t)}\right)\right. \\
&\left. -\ \varOmega^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho\left(\varphi(t), x\right)}{2\varphi\left(T\right)}\right)\right| dydz \\
&+ C_2 \sum_{i=1}^{n} \left\{ \int_0^{|\gamma|}\int_{R^n}\int_{-\infty}^{+\infty} \left|\zeta_i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\left(\varphi_i\left(t,x\right)\right)}{\varphi_i(t)}, \frac{1}{2}\rho'\left(\varphi(t), x\right)\right)\right| \times - \right. \\
&\times \left|K_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho\left(\varphi\left(t,x\right)\right)}{\varphi(t)}\right)\right| \left|\varDelta_i^{m_i}\left(\varphi_i\left(\delta\right)u\right) f\left(x+y+ue_i\right)\right| dydudt \\
&+ \int_{|\gamma|}^{T}\int_{R^n}\int_{-\infty}^{+\infty} \left|K_i^{(\nu)}\left(\frac{y}{\varphi(t)}, \frac{\rho\left(\varphi\left(t,x\right)\right)}{\varphi(t)}\right)\right| \left|\zeta_i\left(\frac{u}{\varphi_i(t)}, \frac{\rho_i\left(\varphi_i\left(t,x\right)\right)}{\varphi_i(t)}, \frac{1}{2}\rho_i'\left(\varphi(t), x\right)\right)\right| \\
&\left. \times \int_0^1 \left|\varDelta_i^{m_i}\left(\varphi_i\left(\delta\right)u\right) f\left(x+y+v\gamma\right)\right| dvdudydt \right\}. \\
&= C_1 A\left(x,\gamma\right) + C_2 \sum_{i=1}^{n} \left(E\left(x,\gamma\right) + F\left(x,\gamma\right)\right), \qquad (2.10)
\end{aligned}$$

where $0 < T \le \{1, T_0\}$. Additionally, we assume that $|\gamma| < T$. Consequently, $|\gamma| < \min\left(\omega, T\right)$. If $x \in G \setminus G_\omega$, then

$$\varDelta\left(\gamma, G\right) D^\nu f\left(x\right) = 0.$$

By inequality (2.9) we have

$$\begin{aligned}
\left\|\varDelta\left(\gamma, G\right) D^\nu f\right\|_{q,G} &\le \left\|A\left(\cdot,\gamma\right)\right\|_{q,G_\omega} \\
&+ \sum_{i=1}^{n} \left(\left\|E\left(\cdot,\gamma\right)\right\|_{q,G_\omega} + \left\|F\left(\cdot,\gamma\right)\right\|_{q,G_\omega}\right), \qquad (2.11)
\end{aligned}$$

$$\begin{aligned}
A\left(x,\gamma\right) \le \prod_{j=1}^{n} \left(\varphi_j(t)\right)^{-\nu_j-2} \int_0^{|\gamma|} d\zeta \int_{R^n}\int_{R^n} \left|f\left(x+\zeta e_\gamma + y\right)\right| \\
\times \left|D_j \varOmega^{(\nu)}\left(\frac{y}{\varphi\left(T\right)}, \frac{\rho\left(\varphi(t), x\right)}{2\varphi(t)}\right) \varOmega^{(\nu)}\left(\frac{z}{\varphi\left(T\right)}, \frac{\rho\left(\varphi(t), x\right)}{2\varphi(t)}\right)\right| dydz.
\end{aligned}$$

Taking into account $\xi e_\gamma + G_\omega \subset G$, and applying the generalized Minkowski inequality, from inequality (1.11) for $U = G$, we have

$$\left\|A\left(\cdot,\gamma\right)\right\|_{q,G_\omega} \le C_1 \left|\gamma\right| \left\|f\right\|_{p,\varphi,\beta;G}. \qquad (2.12)$$

By means of inequality (1.9), for $U = G$, $\eta = |\gamma|$ we get

$$\left\|E\left(\cdot,\gamma\right)\right\|_{q,G_\omega} \le C_2 \left|Q^i_{|\gamma|}\right| \left\|\left(\varphi_i(t)\right)^{-l_i} \varDelta_i^{m_i}\left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G} \qquad (2.13)$$

and by means of inequality (1.10) for $U = G$, $\eta = |\gamma|$ we get

$$\|F(\cdot,\gamma)\|_{q,G_\omega} \le C_3 \left|Q^i_{|\gamma|,T}\right| \left\|(\varphi_i(t))^{-l_i} \Delta_i^{m_i} \left(\varphi_i(t), G_{\varphi(t)}\right) f\right\|_{p,\varphi,\beta;G}. \tag{2.14}$$

From inequalities (2.11) and (2.12)-(2.14) we get the required inequality.
Now suppose that $|\gamma| \ge \min(\omega, T)$. Then

$$\|\Delta(\gamma, G) D^\nu f\|_{q,G} \le 2\|D^\nu f\|_{q,G} \le C(\omega T)\|D^\nu f\|_{q,G} |h(|\gamma|, \varphi; T)|.$$

Estimating for $\|D^\nu f\|_{q,G}$ by means of inequality (2.1), in this case we get estimation (2.8). □

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