Bayes and Invariante Estimation of Parameters Under a Bounded Asymmetric Loss Function

N. Sanjari Farsipour

Department of Statistics, Faculty of Mathematical sciences, Alzahra University, Tehran, Iran
Email: sanjari_n@yahoo.com

Abstract. In this paper we consider the estimation of parameters under a bounded asymmetric loss function. The Bayes and invariant estimator of location and scale parameters in the presence and absence of a nuisance parameter is considered. Some examples in this regard are included.

Keywords: Bayes estimation; invariance; location parameter; scale parameter; bounded Asymmetric loss.

1 Introduction

In the literature, the estimation of a parameter is usually considered when the loss is squared error or in general any convex and symmetric function. The quadratic loss function has been criticized by some researches (e.g., [4], [5], [6] and [7]). The proposed loss function is

$$L(\delta,\theta) = k\{1 - e^{b\{1+a(\delta-\theta)-e^{a(\delta-\theta)}\}}\} \tag{1.1}$$

where $a \neq 0$ determines the shape of the loss function, $b > 0$ serves to scale the loss and $k > 0$ is the maximum loss parameter. The general form of the loss function is illustrated in Figure 1. This is obviously a bounded asymmetric loss function.

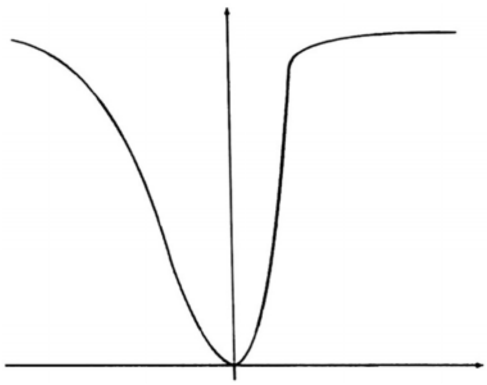

Figure 1. The loss function (1.1) for a=1.

In this paper, we first study the problem of estimation of a location parameter, using the loss function (1.1). In section 2 we introduce the best location-invariant estimator of θ under the loss (1.1). In section 3, Bayesian estimation of the normal mean is obtained under the loss (1.1). Then we study the problem of estimation of a scale parameter, using the loss function

$$L(\delta,\tau) = k\{1 - e^{b\{1+a(\frac{\delta}{\tau}-1)-e^{a(\frac{\delta}{\tau}-1)}\}}\} \tag{1.2}$$

where $a \neq 0, b,k > 0$. The loss (1.2) is scale invariant and bounded. In section 4 we introduce the best invariant estimator of the scale parameter τ under the loss (1.2). Finally in section 5 we consider a subclass of the exponential family and obtain the Bayes estimates of τ under the loss (1.2). Since the parameters b and k do not have any influence on our results, so without loss of generality we take $b = k = 1$ in the rest of the paper.

2 Best Location-Invariant Estimator

Let $\mathbf{X} = (X_1,...,X_n)$ have a joint distribution with probability density $f(\mathbf{X}-\theta) = f(X_1-\theta,...,X_n-\theta)$ where f is known and θ is an unknown location parameter. The class of all location invariant estimators of a location parameter θ is of the form [3]

$$\delta(\mathbf{X}) = \delta_0(\mathbf{X}) - v(\mathbf{Y})$$

where δ_0 is any location-invariant estimator and $\mathbf{Y} = (Y_1,...,Y_{n-1})$ with $Y_i = X_i - X_n$, $i = 1,...,n-1$ and the best location-invariant estimator δ^* of θ under the loss function(1.1), is $\delta^*(\mathbf{X}) = \delta_0(\mathbf{X}) - v^*(\mathbf{y})$, where $v^*(\mathbf{y})$ is a number which minimizes

$$E_{\theta=0}\left[1 - e^{1+a\,(\delta_0(\mathbf{X})-v(\mathbf{y})) - e^{a\,(\delta_0(\mathbf{X})-v(\mathbf{y}))}} \,\Big|\, \mathbf{Y}=\mathbf{y}\right]$$

(see [3]). Differentiating with respect to $v(\mathbf{y})$ and equating to zero, it can be seen that $v^*(\mathbf{y})$ must satisfy the following equation

$$E_{\theta=0}\left[\left(e^{a(\delta_0(\mathbf{X})-v^*(\mathbf{y}))} - 1\right) e^{a(\delta_0(\mathbf{X})-v^*(\mathbf{y})) - e^{a(\delta_0(\mathbf{X})-v^*(\mathbf{y}))}} \,\Big|\, \mathbf{Y}=\mathbf{y}\right] = 0 \qquad (2.1)$$

Example 2.1: (normal mean) Let $X_1,...,X_n$ be i.i.d. random variables having normal distribution with mean θ (real but unknown) and known variance σ^2. If $\delta_0(\mathbf{X}) = \bar{X}$, it follows from Basu's theorem that $\delta_0(\mathbf{X})$ is independent of $\mathbf{Y}$ and hence the best location-invariant estimator of θ is given by $\delta^*(\mathbf{X}) = \bar{X} - v^*$, when v^* is a number which satisfies (2.1), i.e.

$$\int_{-\infty}^{\infty} e^{-\frac{n}{2\sigma^2}(x-\frac{2a\sigma^2}{n})^2 - e^{ax-av^*}}\,\mathrm{d}x = e^{av^* - \frac{a^2\sigma^2}{n}} \int_{-\infty}^{\infty} e^{-\frac{n}{2\sigma^2}(x-\frac{a\sigma^2}{n})^2 - e^{ax-av^*}}\,\mathrm{d}x \qquad (2.2)$$

So, we can find v^* by a numerical solution.

Example 2.2: (Uniform) Let $X_1,...,X_n$ be i.i.d. according to the uniform distribution on $\left(\theta-\frac{\beta}{2},\theta+\frac{\beta}{2}\right)$ where θ is real (but unknown) and $\beta(>0)$ is known. Taking $\delta_0(\mathbf{X}) = (X_{(1)}+X_{(2)})/2$ which is an invariant estimator of θ, the conditional distribution of $\delta_0(\mathbf{X})$ given $\mathbf{Y}=\mathbf{y}$ depends on $\mathbf{y}$ only through differences $X_{(i)} - X_{(1)} = V_i, i = 2,...,n$. Now, note that $\left(X_{(1)}, X_{(n)}\right)$ is a complete sufficient statistic for $\left(\theta,\beta\right)$ and is independent of $Z_i = \frac{X_{(i)}-X_{(1)}}{X_{(n)}-X_{(1)}}$, $i = 2,...,n-1$ for all θ,β by Basu's theorem. Hence $\left(X_{(1)}, X_{(n)}\right)$ and $Z_i^{'}s$ are independent for all θ and any given β. Also, note that the conditional distribution of $\delta_0(\mathbf{X})$ given $V_i^{'}s$ which is equivalent to conditional distribution of $\delta_0(\mathbf{X})$ given $X_{(n)} - X_{(1)}$ and $Z_i^{'}s$ depends only on $X_{(n)} - X_{(1)}$. On the other hand, the conditional distribution of $\delta_0(\mathbf{X})$ given $W = X_{(n)} - X_{(1)}$ at $\theta = 0$ is of the form

$$f_{\delta_0(\mathbf{X})\,|\,W=w}(t) = \frac{1}{\beta-w} \qquad \text{if } |t| < \frac{\beta-w}{2};\ \beta > w$$

Hence the estimator $\delta^*(\mathbf{X}) = \frac{X_{(1)} + X_{(n)}}{2} - v^*$ is the MRE estimator of θ, if ν^* satisfies (2.1), which simplifies to

$$e^{-a(\frac{\beta - w}{2} + v^*) - e^{-a(\frac{\beta - w}{2} + v^*)}} - e^{a(\frac{\beta - w}{2} - v^*) - e^{a(\frac{\beta - w}{2} - v^*)}} = (1+a)\, e^{-e^{a(\frac{\beta - w}{2} - v^*)}} - (1+a)\, e^{-e^{-a(\frac{\beta - w}{2} + v^*)}} \tag{2.3}$$

So, we can find v^* by a numerical solution.

Example 2.3: (Exponential distribution) Let $X_1, \ldots, X_n$ be *i.i.d.* random variables with the density

$$f_\theta(x) = \frac{1}{\beta} e^{-(x-\theta)/\beta} \qquad x \ge \theta$$

where $\theta \in R$ is unknown and $\beta (> 0)$ is known. $\delta_0(\mathbf{X}) = X_{(1)}$ is an equivariant estimator and by the Basu's theorem, it is independent of $\mathbf{Y}$. Therefore, $\delta^*(\mathbf{X}) = X_{(1)} - \nu^*$ is the MRE estimator of θ, if ν^* satisfies (2.1), i.e. satisfies

$$\int_0^{e^{-a\nu^*}} x^{1 - \frac{n}{a\beta}} e^{-x} \, \mathrm{d}x = e^{\frac{n\nu^*}{\beta}} \int_0^{e^{-a\nu^*}} x^{-\frac{n}{a\beta}} e^{-x} \, \mathrm{d}x \qquad ; a < 0$$

$$\int_{e^{-a\nu^*}}^{\infty} x^{1 - \frac{n}{a\beta}} e^{-x} \, \mathrm{d}x = e^{\frac{n\nu^*}{\beta}} \int_{e^{-a\nu^*}}^{\infty} x^{-\frac{n}{a\beta}} e^{-x} \, \mathrm{d}x \qquad ; a > 0$$

which simplifies to

$$\sum_{r=0}^{1-\frac{n}{a\beta}} \frac{(1 - \frac{n}{a\beta})!}{(1 - \frac{n}{a\beta} - r)!} e^{a\nu^*(1 - \frac{n}{a\beta} - r)} = e^{\frac{a\nu^*}{\beta}} \sum_{r=0}^{-\frac{n}{a\beta}} \frac{(-\frac{n}{a\beta})!}{(-\frac{n}{a\beta} - r)!} e^{a\nu^*(\frac{n}{a\beta} + r)} \tag{2.4}$$

So, we can find ν^* by a numerical solution.

3 Bayes Estimation of the Normal Mean

Let $X_1, \ldots, X_n$ be a random sample of size n from a normal distribution with mean θ (unknown parameter) and variance σ^2 (known parameter). In this section we consider Bayesian estimation of the parameter θ using the loss function (1.1).

If the conjugate family of prior distributions for θ is the family normal distributions $N(\mu, b^2)$, then the posterior distribution of θ is $N(m, \upsilon)$ where

$$m = \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu}{b^2}}{\frac{n}{\sigma^2} + \frac{1}{b^2}} \qquad \& \qquad \upsilon = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{b^2}},$$

and the posterior risk of an estimator $\delta(\mathbf{X})$ under the loss function (1.1) is

$$\left\{1 - E\left[e^{1+a(\theta - \delta(\mathbf{X})) - e^{a(\theta - \delta(\mathbf{X}))}} \middle| \mathbf{X}\right]\right\} = 1 - \int_{-\infty}^{\infty} e^{1+a(\theta - \delta(\mathbf{X})) - e^{a(\theta - \delta(\mathbf{X}))}} \frac{1}{\sqrt{n\pi\upsilon}} e^{-\frac{1}{2\upsilon}(\theta - m)^2} \, \mathrm{d}\theta$$

so, $\delta_B(\mathbf{X})$ is the solution of the following integral equation

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2\upsilon}(\theta - 2a\upsilon - m)^2 - e^{a(\theta - \delta_B)}} \, \mathrm{d}\theta = e^{a(\delta_B - a\upsilon - m)} \int_{-\infty}^{\infty} e^{-\frac{1}{2\upsilon}(\theta - a\upsilon - m)^2 - e^{a(\theta - \delta_B)}} \, \mathrm{d}\theta \tag{3.1}$$

Hence, we can find δ_B from the equation (3.1) by a numerical solution.

Also, notice that the generalized Bayes estimator relative to a diffuse prior, $\pi(\theta)=1$ for all $\theta \in R$ can be found by letting $b \to \infty$, i.e. $\upsilon \to \dfrac{\sigma^2}{n}$.

In the presence of a nuisance parameter σ^2, i.e. when σ^2 is unknown, a modified loss function is as follows

$$L(\delta;\theta,\sigma) = 1 - e^{1+a\left(\frac{\delta-\theta}{\sigma}\right) - e^{a\left(\frac{\delta-\theta}{\sigma}\right)}} \tag{3.2}$$

$a \neq 0$ which is a location scale invariant loss function.

In this position, we obtain a class of Bayes estimators of the location parameter θ. Let $\tau = \dfrac{1}{\sigma^2}$ be the precision which is unknown and suppose that conditional on τ, θ has a normal distribution with mean μ and variance $1/\lambda\tau$, where $\mu \in R, \lambda > 0$ are both known constants, i.e., $\theta \mid \tau \sim N\left(\mu, \dfrac{1}{\lambda\tau}\right)$ and τ has a *p.d.f* $g(\tau)$. In this case, one can easily verify that

$$\pi\left(\theta \middle| \mathbf{x},\tau\right) \propto e^{-\frac{r}{2}\sum_{i=1}^{n}(x_i-\theta)^2} e^{-\frac{r\lambda}{2}(\theta-\mu)^2}$$

Or

$$\pi\left(\theta \middle| \mathbf{x},\tau\right) \propto \exp\left\{-\frac{\tau}{2}\left(n+\lambda\right)\left[\theta - \left(\frac{n}{n+\lambda}\bar{x} + \frac{\lambda}{n+\lambda}\mu\right)\right]^2\right\}$$

It is clear that $\theta \mid \mathbf{x},\tau \sim N\left(\eta, \dfrac{1}{\tau(n+\lambda)}\right)$, with $\eta = \dfrac{n}{n+\lambda}\bar{x} + \dfrac{\lambda}{n+\lambda}\mu$. To obtain the Bayes estimate of θ for our problem, it is enough to find an estimate $\delta(x)$ which minimizes $E\left[L\left(\delta(\mathbf{X});\theta,\tau\right)\middle|\mathrm{X},\tau\right]$ for any X,τ. This expectation is under the distribution of $\theta\middle|\mathrm{X},\tau$. So δ_B is the solution of the following integral equation

$$\int_0^\infty \int_{-\infty}^\infty e^{a\sqrt{\tau}\left(\theta-\delta_B\right) - e^{a\sqrt{\tau}(\theta-\delta_B) - \frac{\tau}{2}(n+\lambda)(\theta-\eta)^2}} g(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau = \int_0^\infty \int_{-\infty}^\infty e^{2a\sqrt{\tau}\left(\theta-\delta_B\right) - e^{a\sqrt{\tau}(\theta-\delta_B) - \frac{\tau}{2}(n+\lambda)(\theta-\eta)^2}} g(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau \tag{3.3}$$

which can be solved numerically.

4 Best Scale Invariant Estimator

Consider a random sample $X_1,\ldots,X_n$ from $\dfrac{1}{\tau}f(\dfrac{\mathbf{x}}{\tau})$, where f is a known function, and τ is an unknown scale parameter. It is desired to estimate τ under the loss function (1.2). The class of all scale-invariant estimators of τ is of the form

$$\delta(\mathbf{X}) = \delta_0(\mathbf{X})\big/ W(\mathbf{Z})$$

where δ_0 is any scale-invariant estimator, $\mathbf{X} = (X_1,\ldots,X_n)$, and $\mathbf{Z} = (Z_1,\ldots,Z_n)$ with $Z_i = \dfrac{X_i}{X_n}$; $i = 1,\ldots,n-1, Z_n = \dfrac{X_n}{\left|X_n\right|}$. Moreover the best scale-invariant (minimum risk equivariant (MRE)) estimator δ^* of τ is given by

$$\delta^*(\mathbf{X}) = \delta_0(\mathbf{X})\big/ w^*(\mathbf{Z})$$

where $w^*(\mathbf{Z})$ is a function of $\mathbf{Z}$ which maximizes

$$E_{\tau=1}\left[e^{1+a\left(\frac{\delta_0(\mathrm{X})}{\mathrm{w(Z)}}-1\right)-e^{a\left(\frac{\delta_0(\mathrm{X})}{\mathrm{w(Z)}}-1\right)}} \middle| \mathbf{Z}=\mathbf{z} \right] \tag{4.1}$$

In the presence of a location parameter as a nuisance parameter, the MRE estimator of τ is of the form

$$\delta^*(\mathbf{X})=\delta_0(\mathbf{Y})\big/w^*(\mathbf{Z})$$

where $\delta_0(\mathbf{Y})$ is any finite risk scale-invariant estimator of τ, based on $\mathbf{Y}=(\mathrm{Y}_1,\ldots,\mathrm{Y}_{n-1})$, with $Y_i=X_i-X_n; i=1,\ldots,n-1$, $\mathbf{Z}=(Z_1,\ldots,Z_{n-1}), Z_i=\frac{Y_i}{Y_{n-1}}; i=1,\ldots,n-2$, and $Z_{n-1}=\frac{Y_{n-1}}{|Y_{n-1}|}$ and $w^*(\mathbf{Z})$ is any function of $\mathbf{Z}$ maximizing

$$E_{\tau=1}\left[e^{1+a\left(\frac{\delta_0(\mathrm{Y})}{\mathrm{w(Z)}}-1\right)-e^{a\left(\frac{\delta_0(\mathrm{Y})}{\mathrm{w(Z)}}-1\right)}} \middle| \mathbf{Z}=\mathbf{z} \right] \tag{4.2}$$

In many cases, when $\tau=1$, we can find an equivariant estimator $\delta_0(\mathbf{X})$ or $\delta_0(\mathbf{Y})$ which has the gamma distribution with known parameters ν,η and is independent of $\mathbf{Z}$.

It follows that $\delta^*=\frac{\delta_0}{w^*}$ is the MRE estimator of τ where w^* is a number which maximizes

$$g(w)=\int_0^\infty e^{1+a(\frac{x}{w}-1)-e^{a(\frac{x}{w}-1)}}\frac{\eta^\nu x^{\nu-1}}{\Gamma(\nu)}e^{-\eta x}\,\mathrm{d}\,x=\frac{\eta^\nu}{\Gamma(\nu)}e^{1-a}\int_0^\infty x^{\nu-1}e^{x(\frac{a}{w}-\eta)-e^{a(\frac{x}{w}-1)}}\,\mathrm{d}\,x \tag{4.3}$$

and hence w^* must satisfy the following equation

$$\int_0^\infty x^{\nu-1}e^{(\frac{2a}{w^*}-\eta)x-e^{\frac{ax}{w^*}-a}}\,\mathrm{d}\,x=e^a\int_0^\infty x^\nu e^{(\frac{a}{w^*}-\eta)x-e^{\frac{ax}{w^*}-a}}\,\mathrm{d}\,x \tag{4.4}$$

Theorem 4.1: If $\delta_0(\mathbf{X})$ is a finite risk scale-invariant estimator of τ, which has the gamma distribution with known parameters ν,η, when $\tau=1$. Then the MRE (minimum risk equivariant) estimator of τ under the loss function (1.2) is $\delta^*(\mathbf{X})=\frac{\delta_0(\mathbf{X})}{w^*}$, where w^* must satisfy the equation (4.4).

Example 4.1: (Exponential) Let $X_1,\ldots,X_n$ be a random sample from $E(0,\lambda)$ with density $\frac{1}{\lambda}e^{-\frac{x}{\lambda}}$; $x>0$, and consider the estimation of λ under the loss (1.2). $\delta_0(\mathbf{X})=\sum_{i=1}^n X_i$ is an equivariant estimator which has Ga(n,1)-distribution when $\lambda=1$ and it follows from the Basu's theorem that δ_0 is independent of $\mathbf{Z}$, hence the MRE estimator of λ under the loss (1.2) is $\delta^*(\mathbf{X})=\frac{\sum_{i=1}^n X_i}{\omega^*}$, where ω^* must satisfy the following equation

$$\int_0^\infty x^{n-1}e^{(\frac{2a}{w^*}-1)x-e^{\frac{ax}{w^*}-a}}\,\mathrm{d}\,x=e^a\int_0^\infty x^n e^{(\frac{a}{w^*}-1)x-e^{\frac{ax}{w^*}-a}}\,\mathrm{d}\,x \tag{4.5}$$

Example 4.1: (Continued) Suppose that $X_1,\ldots,X_n$ is a random sample of $E(\theta,\lambda)$ with density $\frac{1}{\lambda}e^{-(x-\theta)/\lambda}$; $x>\theta$, and consider the estimation of λ when θ is unknown. We know that $\left(X_{(1)},\sum_{i=1}^{n}(\mathrm{X}_i-X_{(1)})\right)$ is a complete sufficient statistics for (θ,λ) . It follows that $\delta_0(\mathbf{Y})=2\sum_{i=1}^{n}(X_i-X_{(1)})$ has $\mathrm{Ga}(\mathrm{n}-1,\frac{1}{2})$ -distribution, when $\lambda=1$, and from the Basu's theorem $\delta_0(\mathbf{Y})$ is independent of $\mathbf{Z}$ and hence $\delta^*(\mathbf{X})=\frac{\sum_{i=1}^{n}(X_i-X_{(1)})}{\omega^*}$ is the MRE estimator of λ under the loss (1.2), where ω^* must satisfy the following equation

$$\int_0^\infty x^{n-2}e^{(\frac{2a}{w}-\frac{1}{2})x-e^{\frac{ax}{w}-a}}\,\mathrm{d}\,x=e^a\int_0^\infty x^{n-1}e^{(\frac{a}{w}-\frac{1}{2})x-e^{\frac{ax}{w}-a}}\,\mathrm{d}\,x \tag{4.6}$$

Example 4.2: (Normal variance) Let $X_1,\ldots,X_n$ be a random sample of $N(0,\sigma^2)$ and consider the estimation of σ^2 . $\delta_0(\mathbf{X})=\sum_{i=1}^{n}X_i^2$ is a finite risk scale-invariant estimator of σ^2 and is independent of $\mathbf{Z}$, and when $\sigma^2=1$, $\delta_0(\mathbf{X})$ has $\mathrm{Ga}(\frac{n}{2},\frac{1}{2})$ -distribution and hence $\delta^*(\mathbf{X})=\frac{\sum_{i=1}^{n}X_i^2}{\omega^*}$ is the MRE estimator of σ^2 , where ω^* must satisfy the following equation

$$\int_0^\infty x^{\frac{n}{2}-1}e^{(\frac{2a}{w^*}-\frac{1}{2})x-e^{\frac{ax}{w^*}-a}}\,\mathrm{d}\,x=e^a\int_0^\infty x^{\frac{n}{2}}e^{(\frac{a}{w^*}-\frac{1}{2})\ -e^{\frac{ax}{w^*}-a}}\,\mathrm{d}\,x \tag{4.7}$$

Example 4.2: (Continued) Let $X_1,\ldots,X_n$ be a random sample from $N(\mu,\sigma^2)$, with a nuisance parameter μ . In estimating σ^2 using the loss (1.2), it follows that $\delta_0(\mathbf{X})=\sum_{i=1}^{n}(X_i-\bar{X})^2$ is independent of $\mathbf{Z}$, and when $\sigma^2=1$, the distribution of $\delta_0(\mathbf{Y})$ is $\mathrm{Ga}(\frac{n-1}{2},\frac{1}{2})$. Therefore, $\delta^*(\mathbf{X})=\frac{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}{\omega^*}$ is the MRE estimator of σ^2 , where ω^* must satisfy the following equation

$$\int_0^\infty x^{\frac{n-3}{2}}e^{(\frac{2a}{w^*}-\frac{1}{2})x-e^{\frac{ax}{w^*}-a}}\,\mathrm{d}\,x=e^a\int_0^\infty x^{\frac{n-1}{2}}e^{(\frac{a}{w^*}-\frac{1}{2})x-e^{\frac{ax}{w^*}-a}}\,\mathrm{d}\,x \tag{4.8}$$

Example 4.3: (Inverse Gaussian with zero drift) Let $X_1,\ldots,X_n$ be a random sample of $\mathrm{IG}(\infty,\lambda)$ with density

$$f(x\mid\lambda)=\left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}}e^{-\frac{\lambda}{2x}} \qquad \text{if } x>0$$

and consider the estimation of λ . $\delta_0(\mathbf{X})=\sum_{i=1}^{n}X_i^{-1}$ has $\mathrm{Ga}(\frac{n}{2},\frac{1}{2})$-distribution and is independent of $\mathbf{Z}$ and hence $\delta^*(\mathbf{X})=\frac{\sum_{i=1}^{n}X_i^{-1}}{\omega^*}$ is the MRE estimator of λ , where ω^* must satisfy the equation (4.7).

5 Bayes Estimation of Scale Parameters

In the section, we consider the Bayesian estimation of the scale parameter τ in a subclass of one-parameter exponential families in which the complete sufficient statistic $\delta_0(\mathbf{X})$ has $\mathrm{G}(\nu, \frac{\eta}{2})$ -distribution, where $\nu > 0$, $\eta > 0$ are known.

Assume that the conjugate family of prior distributions for $\beta = \frac{1}{\tau}$ is the family of Gamma distribution $\mathrm{Ga}(\alpha, \xi)$. Now, the posterior distribution of β is $\mathrm{Ga}(\nu + \alpha, \xi + \eta\delta_0(\mathrm{x}))$ and the Bayes estimate of τ is a function $\delta(\mathrm{x})$ which maximizes the function

$$E\left[e^{1+a(\beta\delta-1)-e^{a(\beta\delta-1)}}\middle|\mathbf{X}\right] = \frac{(\eta\delta_0(\mathbf{X})+\xi)^{\nu+\alpha}}{\Gamma(\nu+\alpha)}e^{1-a}\int_0^\infty \beta^{\nu+\alpha-1}e^{(a\delta-\xi-\eta_0(\mathbf{X}))\beta-e^{a(\beta\delta-1)}}\,\mathrm{d}\beta$$

Hence, the maximized δ must satisfy the following integral equation,

$$\int_0^\infty \beta^{\nu+\alpha}e^{(2\mathrm{a}\delta-\xi-\eta\ \delta_0(\mathrm{x}))\beta-e^{a(\beta\delta-1)}}\,\mathrm{d}\beta = e^a\int_0^\infty \beta^{\nu+\alpha}e^{(\mathrm{a}\delta-\xi-\eta\ \delta_0(\mathrm{x}))\beta-e^{a(\beta\delta-1)}}\,\mathrm{d}\beta \quad (5.1)$$

So all estimators satisfying (5.1) are also Bayes estimators.

Example 5.1: (Fisher Nile's problem) The classical example of an ancillary statistic is known as the problem of Nile, originally formulated by Fisher [1]. Assume that X and Y are two positive valued random variables with the joint density function

$$f(x,y;\tau) = e^{-(\tau x + \frac{1}{\tau}y)} \qquad ; x > 0, y > 0, \tau > 0$$

and that (X_i, Y_i), $i = 1,...,n$ is a random sample of n paired observation on (X, Y). Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$, $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$, $T = \sqrt{\frac{\bar{Y}}{\bar{X}}}$, $u = \sqrt{\bar{X}\bar{Y}}$. T is the MLE of τ and the pair (T,U) is a jointly sufficient, but not complete statistics for τ and U is ancillary. Consider a nonrandomized rule $\delta(T,U)$ based on the sufficient statistic $(\bar{X}, \bar{Y})$ which is equivariant under the transformation

$$\begin{pmatrix} z \\ \omega \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & \frac{1}{c} \end{pmatrix}\begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} \quad ; \ c > 0$$

For $\delta(T,U)$ to be scale equivariant, we must have

$$c\delta(T,U) = \delta(\mathrm{c}T,U) \quad ; \quad \forall c > 0 \quad (5.2)$$

Following Lehman [3] a necessary and sufficient condition for an estimator δ to be scale equivariant is that it is of the form $\delta = \delta_0 Z$, where δ_0 satisfies (5.2), hence $\delta_0 = T$, $Z = \phi(U)$. We see that all the scale equivariant estimators $\delta(T,U)$ must have the form

$$\delta(T,U) = T\phi(U) \quad (5.3)$$

using the loss function (1.2) and the fact that the joint distribution of $\left(\frac{T}{\tau}, \mathrm{U}\right)$ is independent of τ, and we can evaluate the risk at $\tau = 1$. Hence

$$R(\tau, T\phi(U)) = E_U[E(1 - e^{1+a\ (T\phi(U)-1)-e^{a\ (T\phi(U)-1)}}) \mid U]$$

It follows that $R(\tau, T(\phi(U))$ is minimized by minimizing the inner expectation. Hence, the minimum risk scale equivariant estimator is $\hat{\tau}_{MRE} = T\phi^*(U)$, where $\phi^*(U)$ must satisfy the following integral equation

$$\int_0^\infty e^{(2\,\mathrm{a}\,\phi^*(\mathrm{u})-\mathrm{u})\,t-\frac{u}{t}-e^{a(\mathrm{t}\,\phi^*(u)-1)}}\,\mathrm{d}\,t = e^a\int_0^\infty e^{(\mathrm{a}\,\phi^*(\mathrm{u})-\mathrm{u})\,t-\frac{u}{t}-e^{a(\mathrm{t}\,\phi^*(u)-1)}}\,\mathrm{d}\,t \tag{5.4}$$

where we use the fact that the joint density function of (T, U) is g(t,u), when $t = 1$, and [2]

$$g(t,\frac{u}{\tau}) = \begin{cases} \dfrac{2e^{-n\,u(\frac{t}{\tau}+\frac{\tau}{t})}u^{2n-1}}{n^{-2n}[(n-1)!]^2\,t} & \text{if} \quad t > 0, \mathrm{u} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

For deriving the Bayes estimator of τ, let us consider the Inverted Gamma distribution as a prior distribution

$$\pi_{\alpha,\lambda}(\tau) = \frac{\lambda^\alpha e^{-\lambda/\tau}}{\tau^{\alpha+1}\Gamma(\alpha)} \quad ; \quad \tau > 0 \ , \ \lambda > 0.$$

Therefore the unique Bayes estimator δ_B which is admissible under the loss (1.2) must satisfy the following integral equation

$$\int_0^\infty \tau^{-\alpha} e^{(2\,\mathrm{a}\,\delta_B-\frac{u}{t})\,\tau-(\lambda+u\,t)\frac{1}{\tau}-e^{a(\tau\,\delta_B-1)}}\,\mathrm{d}\,\tau = e^a\int_0^\infty \tau^{-\alpha} e^{(\mathrm{a}\,\delta_B-\frac{u}{t})\,\tau-(\lambda+u\,t)\frac{1}{\tau}-e^{a(\tau\,\delta_B-1)}}\,\mathrm{d}\,\tau \tag{5.5}$$

Note that $\hat{\tau}_{MRE} = \hat{\tau}_B$, whenever $\alpha \to 0$, $\lambda \to 0$. This means that when the loss function is scale invariant loss (1.2), then $\hat{\tau}_{MRE}$ is a generalized Byes rule against the scale invariant improper prior $\pi(\tau) = \dfrac{1}{\tau}$; $\tau > 0$ and is therefore minimax.

References

1. Fisher, R. A. (1959): Statistical Method and Scientific Inference. London, Oliver and Boyd.
2. Joshi, s. and Nabar, S. (1987): Estimation of the Parameter in the Problem of the Nile. Commun. Statist. Theory Meth. 16, 3149-3156.
3. Lehman, E.L. and Casella, G. (1998): Theory of Point Estimation. Springer-Varlag New York, Inc.
4. Leon, R. V. and Wu, C.F.G. (1992): A Theory of Performance Measure in Parameter Design. Statist. Sinica, 2(2), 335-357.
5. Tribus, M. and Szonya, G. (1989): An Alternate View of the Taguchi Approach. Quality progress, May, 46-52.
6. Varian, H. R. (1975): A Bayesian Approach to Real Assessment. In: Studies in Bayesian Econometric and Statistics in Honor of Leonard J. Savage, eds. S.E. Fienberg and A. Zellner. North Holland, Amesterdam, 195-208.
7. Zellner, A. (1986): Bayesian Estimation and Prediction Using Asymmetric Loss Function. J. Amer Statist. Assoc., 81,446-451.

