Solution of Diffusive Wave Equation Using FEM for Flood Forecasting

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ABSTRACT

Flood forecasting can be predicted numerically by the solution of 1D (One-Dimensional) unsteady diffusive wave equation. This research paper presents the development of a Finite Element Model for flood routing using diffusive wave equation with and without lateral inflow/outflow. The model is based on two-step semi–implicit Taylor–Galerkin technique. The accuracy of the model has been verified by comparing computed results with available solution of the diffusive wave problems available in open literature. Numerical results demonstrate that the technique is an efficient and accurate tool to simulate diffusive wave equation for outflow hydrograph for temporal variation of lateral inflow and outflow.

Key Words: One-Dimensional, Finite Element Model, Flood Routing, Lateral Inflow and Outflow, Hydrograph.

1. INTRODUCTION

The occurrence of floods globally has increased due to urbanization and growth of population. Pakistan has a long history of flooding from the Indus River and its tributaries due to heavy rainfall mostly occur in the monsoon season (July-September). Due to high floods, loss of human lives, crops and livestock, damages of infrastructures and extensive erosion caused.

The successful research study of any river depends upon the data availability and the tools for its analysis. The best tool to forecast the flood is the use of flood routing technique.

The flood advances in flood analysis include, the solution of the practical problem in the form of probabilistic design value for the engineering structures such as levees, reservoirs, spillways etc.

the behavior of the catchment system, understanding the flood phenomena itself through basic research on the rainfall-runoff process and; the development of forecast models for the flood warning systems [1]. Out of above advances, there is still room for research to be conducted on the last theme, through developing an efficient and accurate numerical model for unsteady flow in order to forecast the flood-warning program.

The one-dimensional, fixed bed water surface profile and steady state are computed as river hydraulic studies. The Flood management plan and flood control research study may need of some consideration of unsteady flow, mobile boundary that can change with time and flow and multi-dimensional characteristics to perform the required studies properly [1].

1D numerical models have extensively been used to compute unsteady free surface flows of tidal oscillating and flows produced by the operation of control structures, the first attempt was made by [2]. Since then, a variety of numerical techniques have been used to solve shallow water wave equations [3-4].

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The finite volume has not presented on a frictionless bed for exact solution for a dam break flood. The realistic case involving a rough bed was addressed by a numbers authors but only approximate solution was developed to date. A non-uniform velocity distribution in the vertical direction leads to numerical difficulties but self-similar solution can still be obtained current with variables inflow [5].

The FDM (Finite Difference Method) is simple and requires less computation for simple problems. FDM proves complicated and less accurate owing to its inflexibility to deal with requisite complex geometrical specification of problems [6-7].

The FEM was applied for solving individual channels and a double sweep method used to save computing time for computation two channel networks by [8]; the obtained results by FEM was very close to those provided by the widely applied Preissmann scheme, showing that FEM is an accurate method to model flows in open channels [8].

In most of numerical models, the FDM has been used due to the fact that the FDM requires using small space and time steps, but which decreases its efficiency. On the contrary, the finite element method is not only accurate and efficient method but also flexible in handling general shapes of domain and boundary conditions, which is vital for fluid dynamic problems that often require the calculation of flows in complex geometrical phenomenon [9-10].

The diffusive wave equation describes both convection and diffusion terms depend on two parameters, celerity and diffusivity, which are functions of the discharge. The resolution of this equation depends also on initial conditions, the input hydrograph, the lateral inflow/outflow hydrographs and geometric characteristics of the channel [11].

The Taylor-Galerkin scheme generates an accurate time-marching technique in accordance with high spatial resolution, which results from a Galerkin approximation [12]. Therefore, in this research, the FEM using two-step predictor-corrector Taylor-Galerkin scheme is used to develop 1D numerical model for the computation of flood routing at the downstream. The developed model is validated through comparing discharge values (outflow hydrograph) with the analytical solution and available numerical predictions of [11].

2. GOVERNING EQUATIONS

Neglecting the local inertia and momentum source terms in the equation of motion in Saint-Venant system, [13-15] leading in the following equation for stream flow and overland flow for the diffusive wave model.

$$\frac{\partial Q}{\partial t} = -C\frac{\partial Q}{\partial x} + D\frac{\partial^2 Q}{\partial x^2} + Cq \qquad (1)$$

Where Q is Discharge, C is Wave celerity, D is Diffusivity, q is Lateral flow, t is time and x is Flow direction.

2.1 Non-Dimensionalization

For convenience, the governing system of equation is cast into following non-dimensional forms: space and time

$$x^* = \frac{Cx}{D}, \qquad t^* = \frac{C^2 t}{D}$$

inflow characteristics

$$Q^* = \frac{Q}{Q_0}$$

Where Q_0 is Initial discharge lateral flow

$$q^* = \frac{qD}{Q_0 C}$$

putting above non-dimensional parameters in Equation (1), we get:

$$\frac{\partial Q^*}{\partial t^*} = -\frac{\partial Q^*}{\partial x^*} + \frac{\partial^2 Q^*}{\partial x^{*2}} + Cq^* \qquad (2)$$

dropping the asterisks for simplicity and brevity, the Equation (2) can be written as:

$$\frac{\partial Q}{\partial t} = -\frac{\partial Q}{\partial x} + \frac{\partial^2 Q}{\partial x^2} + Cq \qquad (3)$$

Where additionally, Q* is non-dimensionalized value of Q, q* is non-dimenisonalized value of q, t* is non-dimenisonalized value of t, and x* is non-dimenisonalized value of x,

2.2 Wave Celerity

A wave is a deviation in a flow, such as a fluctuation in flow rate or water surface elevation, and wave celerity is velocity which develops variation travels along with the channel. It depends on the type of wave being considered and may be quite different from the water velocity. The kinematic wave celerity can be expressed in terms of depth Y as under:

$$\text{Celerity (C)} = \frac{1}{B}\frac{dQ}{dY} \tag{4}$$

Where B is Width of the channel and Y is Depth of water.

2.3 Diffusion Coefficient or Diffusivity

The diffusion coefficient includes the combined effects of molecular diffusion turbulent mixing, and the rise and fall of a flood occurs much more slowly than many other types of unsteady flow changes. It is an important parameter known to govern the 1D flow in streams. It is necessary to estimate the diffusion coefficient of flow in streams and channels, which has been of concern to hydrologists, civil engineers and environmental scientists for the last five decades. As a result, a considerable research has been applied to its conceptual and analytical modeling by different researchers. However, the most appropriate and accurate proposed method for determination of diffusion coefficient for some mighty rivers presented by [16] have recommended following formula.

$$\textit{Diffusion Coefficient} = D = \frac{0.058Q}{BS} \tag{5}$$

where, S is Longitudinal bed slope.

3. NUMERICAL SCHEME

In the proposed finite element scheme two-step Lax-Wendroff predictor-corrector technique has been adopted. The choice of numerical scheme is rest on the accuracy, efficiency and stability of the scheme. Literature shows that in an explicit scheme the basic difficulty arrives when using large time steps (Δt) which leads to use the alternate approach i.e. semi-implicit scheme. Semi-implicit scheme allows calculations to be performed with the large time-steps, which enhance the numerical stability and efficiency [9-10, 17]. The finite element scheme is described in following two steps:

Step 1: Predict the discharge Q at half time step (n+1/2) level using following Equation (6).

$$\left(\frac{2M}{\Delta t}+\frac{S_Q}{2}\right)\left(Q_j^{n+\frac{1}{2}}-Q_j^{n}\right) = -\left(N(C)+D\ S_Q\right)Q_j^{\ n}+M\ q_j^{\ n}+b.t. \tag{6}$$

Step-2: Using the above information, correct the second order accurate Q at full time (n+1) level using following Equation (7).

$$\left(\frac{M}{\Delta t}+\frac{S_Q}{2}\right)\left(Q_j^{n+1}-Q_j^{n}\right) = -N(C)Q_j^{n+\frac{1}{2}}-D\,S_Q\,Q_j^{n}+M\,q_j^{n+\frac{1}{2}}+b.t. \tag{7}$$

where,

$$M = \int_\Omega \varphi_i\varphi_j\,d\Omega = MassMatrix$$

$$N(C) = \int_\Omega \frac{\partial\varphi_i}{\partial x}\varphi_i\varphi_j\,d\Omega = Non-LinearConvectiveMatrix$$

$$S_Q = \int_\Omega\left[\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x}\right]d\Omega = \textit{Diffusive Matrix}$$

b.t = boundary terms

4. PROBLEM SPECIFICATION

A channel having its reach length (L) of 30 units was used for this study. At the upstream, the prescribed inflow hydrograph is the function of parameter of form of Hayami function z, time t, Kernal function θ, and base flow Q_b as under:

$$\text{Upstream Inflow } Q = f\,(z,\ t,\ \theta,\ Q_b) \tag{8}$$

Where, z = a form parameter and θ = time that represents the centre of gravity of hydrograph. For this problem, the upstream boundary condition is described by the following Equation (9):

$$Q^*(0,t^*) = \frac{\exp\left[z^*\left(2-\frac{t^*}{\theta^*}-\frac{\theta^*}{t^*}\right)\right]}{\left(t^*/\theta^*\right)^{3/2}}+Q_b^* \tag{9}$$

where, non-dimensional form parameter z*=2, non-dimensional Kernel function θ^*=30, and non-dimensional base flow Q_b^* = 0.005

Following three cases have been used for the computation of outflow hydrographs:

Case-1: Inflow Hydrograph without Lateral Flow q^* = 0 (Fig. 1).

Case-2: Inflow Hydrograph with Lateral Inflow $q^* = 0.4\ (Q^* - Q_b^*)/L^*$ (Fig. 2).

Case-3: Inflow Hydrograph with Lateral Outflow $q^* = -0.4\ (Q^* - Q_b^*)/L^*$ (Fig.3).

5. RESULTS AND DISCUSSION

Above three test cases are computed using FEM described in Equations (6-7) applying upstream boundary condition given in Equation (9). The predicted outflow hydrographs are compared with available outflow hydrograph given in [11] and are discussed as under:

5.1 Case-1: Inflow Hydrograph without Lateral inflow $q^* = 0$

The outflow hydrograph predicted using the proposed finite element model given in Fig. 4(a) shows good agreement when compared with analytical solution and

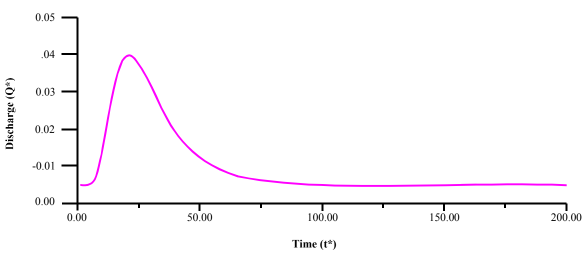


FIG. 1. CASE-1: INFLOW HYDROGRAPH WITHOUT LATERAL INFLOW Q = 0*

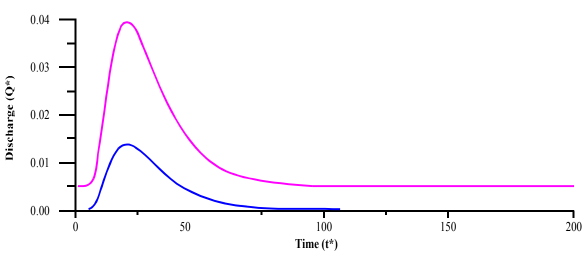


FIG. 2. CASE-2: INFLOW HYDROGRAPH WITH LATERAL INFLOW $Q^* = 0.4\ (Q^* - Q_B^*)/L^*$

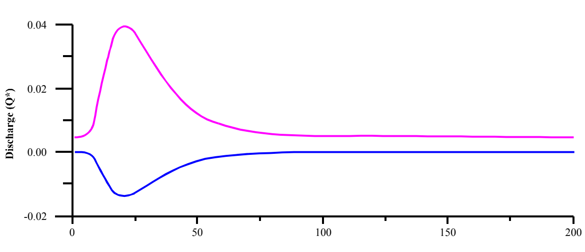


FIG. 3. CASE-3: INFLOW HYDROGRAPH WITH LATERAL OUTFLOW $Q^* = -0.4\ (Q^* - Q_B^*)/L$

outflow hydrograph given in Fig. 4(b) computed using finite difference method by Moussa and Bocquillon [11]. During computation, it is observed that the proposed model is stable and accurate.

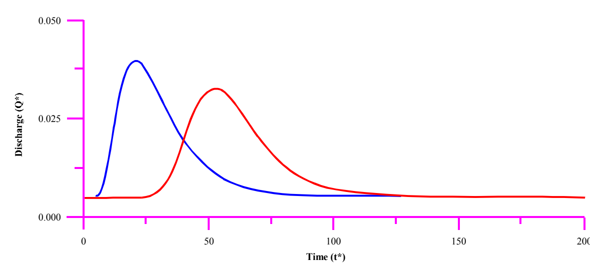
The present model has been tested with various time steps (Δt) of 0.1, 0.5 and 1.0 and element sizes (Δx) of 0.5 and 1.0. It has been observed that the numerical scheme predict the accurate results even increasing both Δt and Δx to 1.0. This show that with such increased time step (Δt) and element size (Δx) reduce the computation cost; Hence, the efficiency of the model is increased.

5.2 Case-2: Inflow Hydrograph with Lateral Inflow

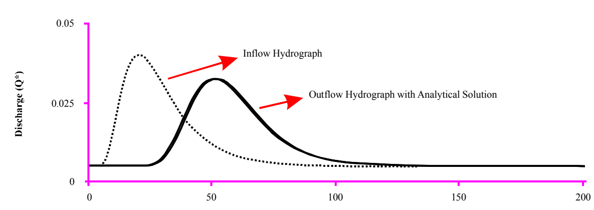
For this case outflow hydrograph was computed with the lateral inflow [q*=0.4 (Q-Qb)/L] described in Fig. 5(a) which is compared with that of [11] and analytical solution given in Fig. 5(b). This comparison shows good resolution between them.

5.3 Case-3: Inflow Hydrograph with Lateral Outflow $q^* = -0.4\ (Q^*-Q_b^*)/L$

In this case, the data of inflow hydrograph and lateral outflow hydrograph of [11] are used in proposed FEM and predicted the outflow hydrograph. The predicted outflow hydrograph in Fig. 6(a) is compared with that of Moussa and Bocquillon [11] described in Fig. 6(b) their comparison shows a nice agreement of simulation between them.



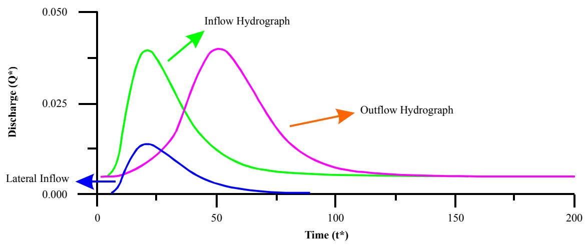
(a) OUTFLOW HYDROGRAPH PREDICTED BY PROPOSED FINITE ELEMENT MODEL



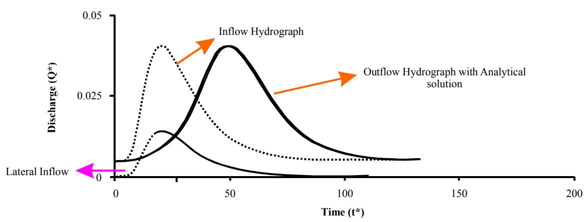
(b) OUTFLOW HYDROGRAPH BY FDM [11] AND ANALYTICAL SOLUTION

FIG. 4. COMPARISON OF PRESENTED FE MODEL WITH ANALYTICAL SOLUTION AND FDM OF [11]

On the whole, it is concluded that the FEM developed model is efficient and accurate for predicting outflow hydrograph not only for given inflow hydrographs but also for lateral inflow and outflow. This developed FEM has also been verified against hydrographs computed by the finite FDM and HEC-RAS for a 29 km river [18-21] and applied with observed discharges at upstream of Sukkur and Kotri barrages of the Indus River [22,23].

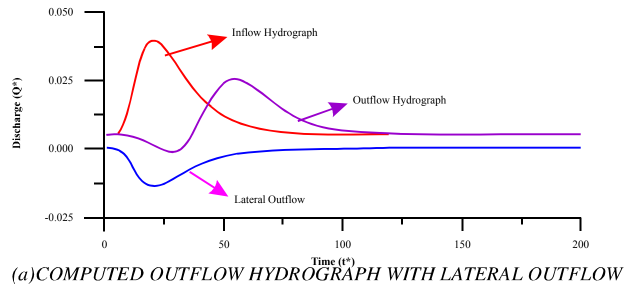


(a) PREDICTED OUTFLOW HYDROGRAPH WITH LATERAL INFLOW FOR THE PRESENT STUDY

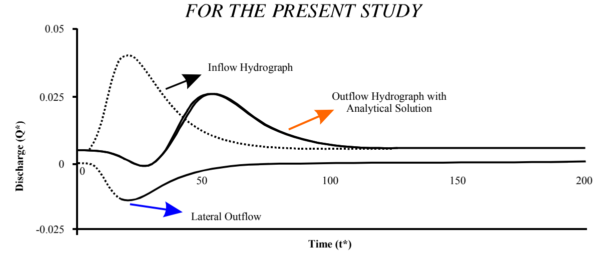


(b) OUTFLOW HYDROGRAPH WITH LATERAL INFLOW [11] AND ANALYTICAL SOLUTION

FIG. 5. COMPARISON OF PRESENT MODEL WITH ANALYTICAL SOLUTION AND FDM [11]



(a)COMPUTED OUTFLOW HYDROGRAPH WITH LATERAL OUTFLOW FOR THE PRESENT STUDY



(b) OUTFLOW HYDROGRAPH WITH LATERAL OUTFLOW [11] AND ANALYTICAL SOLUTION

FIG. 6. COMPARISON OF PRESENT FINITE ELEMENT MODEL WITH ANALYTICAL SOLUTION AND FDM OF [11]

6. CONCLUSIONS

The developed FEM using two-step predictor-corrector Taylor-Galerkin scheme has been applied for the prediction of flood routing. The model shows stability, accuracy and efficiency for the solution of diffusive wave equation with and without lateral inflow/ outflow.

The validation of accuracy of this model was made through comparing the numerical prediction against the analytical solution and numerical results of Moussa and Bocquillon for all three cases on spatial and temporal distribution of lateral flow.

Efficiency and accuracy of this model has been studied by increasing element size (Δx) and time step (Δt) up to 1.0 for predicting outflow hydrograph for flood routing with lateral inflow and outflow hydrographs.

It is suggested that the proposed Finite Element model can be applied for predicting the flood routing in any river by providing suitable Celerity and Diffusivity values and boundary conditions.

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REFERENCES

[1] US Army Corps of Engineers Washington, "River Hydraulic: Engineering and Design", Engineering Manual No. 1110-2-1416, Department of the Army, US Army Corps of Engineers, Washington, DC, 15th October, 1993.

[2] Stoker, J.J., "Numerical Solution of Flood Prediction and River Regulation Problems", Reprint IMM-200, Institute of Mathematics and Science., New York, 1953.

[3] Cunge, J.A., Holly, F.M., and Verway, A., "Practical Aspects of Computational River Hydraulics", Pitman Publishing Ltd., Boston Mass, 1980.

[4] Chaudhry M.H., "Open Channel Flow", Prentice-Hall Inc., Englewood Cliffs, NJ, 1993.

[5] Ancey, C., and Rentschler, M., "An Exact Solution for Dam-Break on Steep Slopes" School of Architecture, Civil and Environmental Engineering, Switzerland, 2007.

[6] Peyret, R., and Taylor, T.D., "Computational Methods for Fluid Flow", Springer-Verlag, New York, 1983.

[7] Hircsh, C., "Numerical Computation of Internal and External Flows, Velocity; Computational Methods for in Viscid and Viscous Flows", John Wiley and Sons, Chichester-Singh, 1990.

[8] Zhang, Y., "Simulation of Open Channel Network Flows Using Finite Element Approach", Basin Analysis Program, Environmental Protection Division, Georgia Department of Natural Resources, Floyd Towers East, Suite 1058, 2 Martin Luther King, Jr. Drive, SE., Atlanta, GA 30334, USA, 2004.

[9] Qureshi, A.L., "Numerical Simulation of Sediment Transport Problems", Ph D Dissertation, Mehran University of Engineering and Technology, Jamshoro, Pakistan, 2004.

[10] Qureshi, A.L., and Baloch, A., "Finite Element Simulation of Sediment Transport: Development, Validation and Application of Model to Photo Minor of Jamrao West Branch, Sindh, Pakistan", Advances in River Sediment Research, Edited by Fukuoka, et al., CRC Press, Taylor and Francis Group, London, UK, 2013.

[11] Moussa, R., and Bocquillon, C., "Fractional-Step Method Solution of Diffusive Wave Equation", Journal of Hydrologic Engineering, ASCE, Volume 6, No. 1, pp 11-19, 2001.

[12] Donea, J., Giuliani, S., Laval, H., and Quartapelle, L., "Time-Accurate Solution of Advection-Diffusion Problems by Finite Element", Computer Methods Applied Mechanical Engineering, Volume 45, pp. 123-145, 1984.

[13] Hayami, S., "On the Propagation of Flood Water", Disaster Presentation,Research Institute of Bulletin, pp. 1-16, Kyoto, Japan, 1951.

[14] Lighthill, M.J., and Whitham, G.M., "Kinematic Waves, I-Flood movement in Long River", Proceedings of Royal Society, London, Services-A, 229, pp. 281-316, 1955.

[15] Henderson, F.M., "Flood Wave in Prismatic Channel" Journal of Hydraulic Division, ASCE, Volume 89, No. 4, pp. 39-67, 1963.

[16] Keefer, T.N., and Mc Quivery, R.S., "Multiple Linearization Flow Routing Method", Journal of Hydraulic Division, ASCE, Volume 100, No. 7, pp. 1031-1046, 1974.

[17] Memon, G.Q., Baloch, A., and Qureshi, A.L., "Analytical and Numerical Solution of Seepage Head from Irrigation Canal in Porous Media", Mehran University Research Journal of Engineering & Technology, Volume 29, No. 2, pp. 211-230, Jamshoro, Pakistan, April, 2010.

[18] Preissmann, A., "Propagation of Translatory Waves in Channel sand Rivers", Proceedings of 1st Congress of French Association for Computation, Grenoble, France, AFCAL, pp. 433-442 (In French), 1961.

[19] Moghaddam, and Firoozi, "Development of Dynamic Flood Wave Routing in Natural Rivers through Implicit Numerical Method", American Journal of Scientific Research, No. 14, pp. 6-17, 2011.

[20] Qureshi, A.L., Mahessar, A.A., and Baloch, A., "Verification and Application of Finite Element Model Developed for Flood Routing in Rivers", International Journal of Environmental, Earth Science and Engineering, WASET, Volume 8, No. 2, pp. 24-27, February, 2014.

[21] Delphi, M., Shooshtari, M.M., and Zadeh, H.H., "Application of Diffusion Wave Method for Flood Routing in Karun River", International Journal of Environmental Science and Development, Volume 1, No. 5, pp. 432-434, December, 2010.

[22] Mahessar, A.A., Qureshi, A.L., and Baloch, A., "Numerical Study of Flood Routing in Indus River", International Water Technology Journal (IWTJ), Volume 3, No. 1, pp. 3-12, March, 2013.

[23] Mahessar, A.A., Qureshi, A.L., and Baloch, A., "Flood Forecasting for the Super Flood 2010 in Sukkur-Kotri Reach of Indus River", International Water Technology Journal (IWTJ), Volume 3, No. 4, pp. 255-262, December, 2013.