FEM based Approximations for the TV Denoising Optimization Problem

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ABSTRACT

FEM (Finite Element Method) based approximation model is proposed in this paper. Our goal is to solve a TV (Total Variation) denoising optimization problem using finite element method with triangular grid as a computational domain and also to study the local choice of smoothness parameters for the given problem. We provide some confidence measures in the form of average absolute error and average absolute square error corresponding to the different choices of locally selected regularization weights; moreover we show that how the regularization parameters play a crucial part in the reduction of error in the approximations and the diffusion enhancement specifically for this TV model.

Key Words: **Image Denoising, TV Regularization, Adaptive Finite Elements, Optimization.**

1. INRODUCTION

Energy based Optimization in the image processing is a famous approach [1-13]. Energy methods for denoising models are usually consisting of the minimization of a suitable energy functional which includes two parts called diffusion part and reaction part. The diffusion part penalizes the signal by its smoothing effect, whereas; the reaction part minimizes the deviations in the original and the noisy image, consequently the low pass filter is obtained. As the Tikhonov regularizer produces the quadratic regularization having extra blurring effects, these regularization effects generally damage the image corners which is a major drawback of this regularization strategy. To cope with such like difficulties and to recover the sharp corners the minimization approaches like Total variation and Perona-Malik [1-2] were suggested. The TV regularization has the properties to recover the Hyper surfaces successfully [1-2,5,7,9,11,14]. In our previous work [2] a novel adaptive strategy for the Perona-Malik diffusion along with the study of individual locally selection of the smoothness parameters was proposed for the given ill-posed problem which creates some ambiguities in the newly introduced automatic

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regularization procedure, the same strategy was applied on Tikhonov regularization approach in [4] but due to the extra regularization effects from Tikhonov regularization we enhance our FEM based work to the famous regularization called TV method. In this work we contribute a study of local smoothness parameters using a FEM based approximating implicit model for the image restoration problem which is one of the well-known problems in the image processing research [7,9,11,12,14-18].

In general practice the finite difference methods are used as computational strategies for partial differential equations in image processing and computer vision research, where as we use FEM based implicit scheme for the approximation of diffusion image from the noisy image using partial differential equation associated to the TV energy model. The approach of using the triangular grids for oscillations reduction in images is novel and flexibly allows the adaptive choice of smoothness without creating any extra damaged regions in the image.

We consider the following optimization problem for the computation of a low frequency image signal u from a noisy image $f:\Omega\rightarrow R$

$$\min_u = \frac{1}{2}\lambda\left\|u-f\right\|^2_{L^2(\Omega)} + \alpha\,\tau v(u) \tag{1}$$

The $\alpha>0$ is uniformly smoothness parameter over the domain Ω which is locally selected and the λ is a small regularization parameter which lies in the interval $0<\lambda<1$.

$$\|u\|_{L^{(2)}(\Omega)} = \left(\int_\Omega |u|^2\right)^{\frac{1}{2}}$$
$$\tau v(u) = \int_\Omega |\nabla u|\, dx \tag{2}$$

To solve the minimization problem (1) we use direct methods from the calculus of variations [16] where we have the following Euler-Lagrange equation associated to the minimization problem Equation (1) is given as:

$$0 = -\mathrm{div}\left(\frac{\alpha\cdot\nabla u}{\sqrt{\beta^2+|\nabla u|^2}}\right) + \lambda(u-f)\, in\,\Omega \tag{3}$$

where, $\frac{\partial u}{\partial n}=0$, on $\partial\Omega$., β is strictly small positive parameter used to avoid the division by zero,. The gradient system associated to steady state problem Equation (3) is:

$$\partial_\rho u = -\mathrm{div}\left(\frac{\alpha\cdot\nabla u}{\sqrt{\beta^2+|\nabla u|^2}}\right) + \lambda(u-f) \tag{4}$$

Here the ρ is considered as an artificial evolution parameter.

2. VARIATIONAL FORMULATION

We choose FEM based implicit numerical approximation model to solve the partial differential Equation (4). To proceed further the variational formulation for Equation (4) is given as:

$$b(u,v) = l(f,v) \tag{5}$$

Where $b(u,v)$ is called bilinear form and $l(f,v)$ linear from. The week problem is to compute the signal u with reduced the noise in a Sobolev space $H^1(\Omega)\subseteq X$ such that:

$$\begin{cases} b(u,v) = \int_\Omega\left(\frac{\alpha\nabla u\cdot\nabla v}{\sqrt{\beta^2+|\nabla u|^2}}\right)dx + \lambda\int_\Omega u\cdot v dx \\ l(u,v) = \lambda\int_\Omega f\cdot v dx \forall v\in H^1\subseteq X \end{cases} \tag{6}$$

H^1 is a Sobolev space which is defined as:

$$H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in (L^2(\Omega))^2\}$$

for further details about Sobolev spaces, we refer the reader to review the basic theory of finite elements and Sobolev spaces in [19-20].

3. DISCRETE PROBLEM

For a triangular domain with a family of elements T_h as triangles with maximum size h>0, the discrete space as $X_h \subset X$, is introduced as

$$X_h := \{v_h \in C^0(\overline{\Omega}) | \forall K \in T_h, v_h|_K \in P_1(K)\}$$

The notation $P_1(K)$shows the FEM based P_1 interpolation functions. The following algorithm for the approximation of the diffused signal is considered as a discrete problem which is a FEM based implicit numerical model.

$$(I + \tau A_\alpha)\, U^{K+1} = U^K + \tau L \qquad (7)$$

Where the time derivative is discretized using forward difference operator and

$$U = [u_1, u_2, ..., u_N] \qquad (8)$$

The $u_1, u_2, ..., u_N$ are the grey values with suppressed noise corresponding to the N number of nodes on the triangular grid. The vector L is obtained from the linear form given as $l(f,v)$. As the bilinear form, given in (6) is symmetric and positive definite. The given implicit numerical scheme is unconditionally stable.

3.1 Absolute Average Square and Absolute Average Differences

To check the efficiency and the influence of regularization weights by locally selected parameters the error is proposed as AASD (Average Absolute Square Difference) and AAD (Average Absolute Difference) between the given original image and the approximate computed diffused image. The formulas for AASD and AAD are given as:

$$AAD = \frac{\sum |u - u_h|}{|\Omega|} \qquad (9)$$

$$AASD = \frac{\sum |u - u_h|^2}{|\Omega|} \qquad (10)$$

Where u is the original signal intensity and u_h is the corresponding approximate diffused grey value. The results for the AAD and AASD for the different selected values of local smoothness parameters are given in the Table 1 and the corresponding Figs. 1-2.

4. RESULTS AND DISCUSSION

We consider a test image House downloaded from http://decsai.ugr.es/~javier/denoised/test_images/index.htm. The first part of computational production is given as quantitative information showing the computational error as AAD and AASD obtained from various adaptive

TABLE 1. AVERAGE AAD AND AASD RESULTS FOR DIFFERENT CHOICES OF SELECTED PARAMETER α

Choices for α	AAD	AASD
100	0.0961	0.7115
50	0.0909	0.6876
25	0.0863	0.6738
	0.0747	0.6394
5	0.0567	0.5864
0.9	0.0274	0.2483
0.5	0.0223	0.1945
0.1	0.0336	0.2952
0.01	0.0356	0.3642
0.001	0.0350	0.3687

uniform choices of smoothness parameters α,β and λ. We considered the value of the β as very small to prevent the approximation procedure from zero division, here for this example we consider the β=0.001as fixed for all readings, similarly the smoothness weight λ=0.01. The regularization effects are then shown with respect to various choices of the α. The convergence results with respect to the uniformly selected α are given in Table 1, Figs. 1-2 and as shown in image results Fig. 3(a-l). From the quantitative information one can easily observe that error decreases with respect to decrease in the smoothness parameter α. The second part of the computational results is the denoised images, one can observe from the denoised images that the noise is successfully decreased as we decrease the parameter α, but one can observe from the results that with very small choice of local parameter α, as we go smaller than 0.5 the level of oscillations in the given solution again

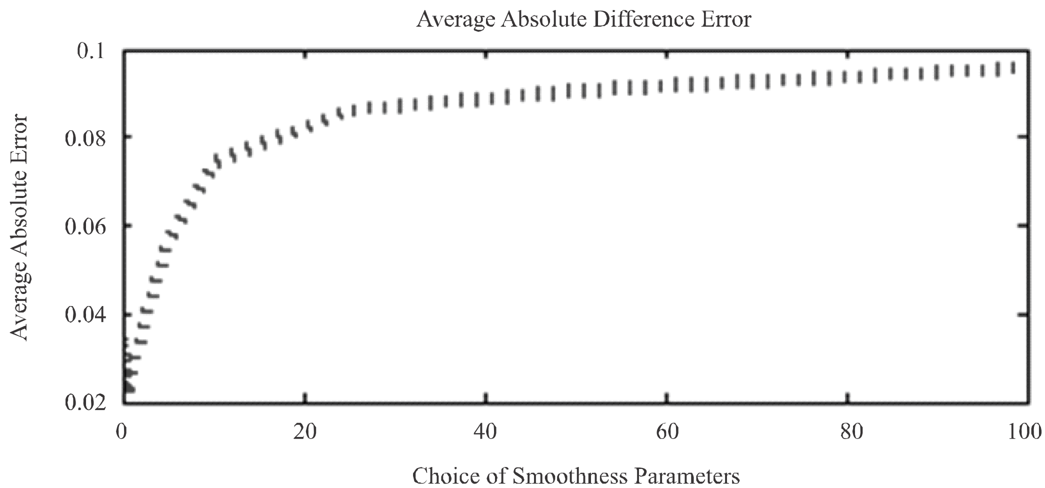


FIG. 1. AVERAGE AAD RESULTS FOR DIFFERENT CHOICES OF SELECTED PARAMETER α

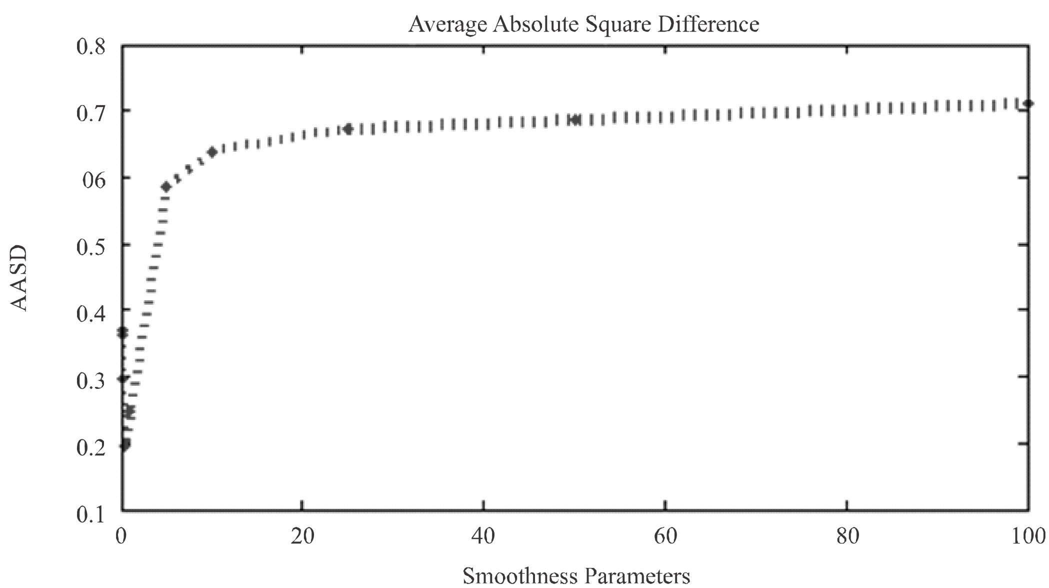
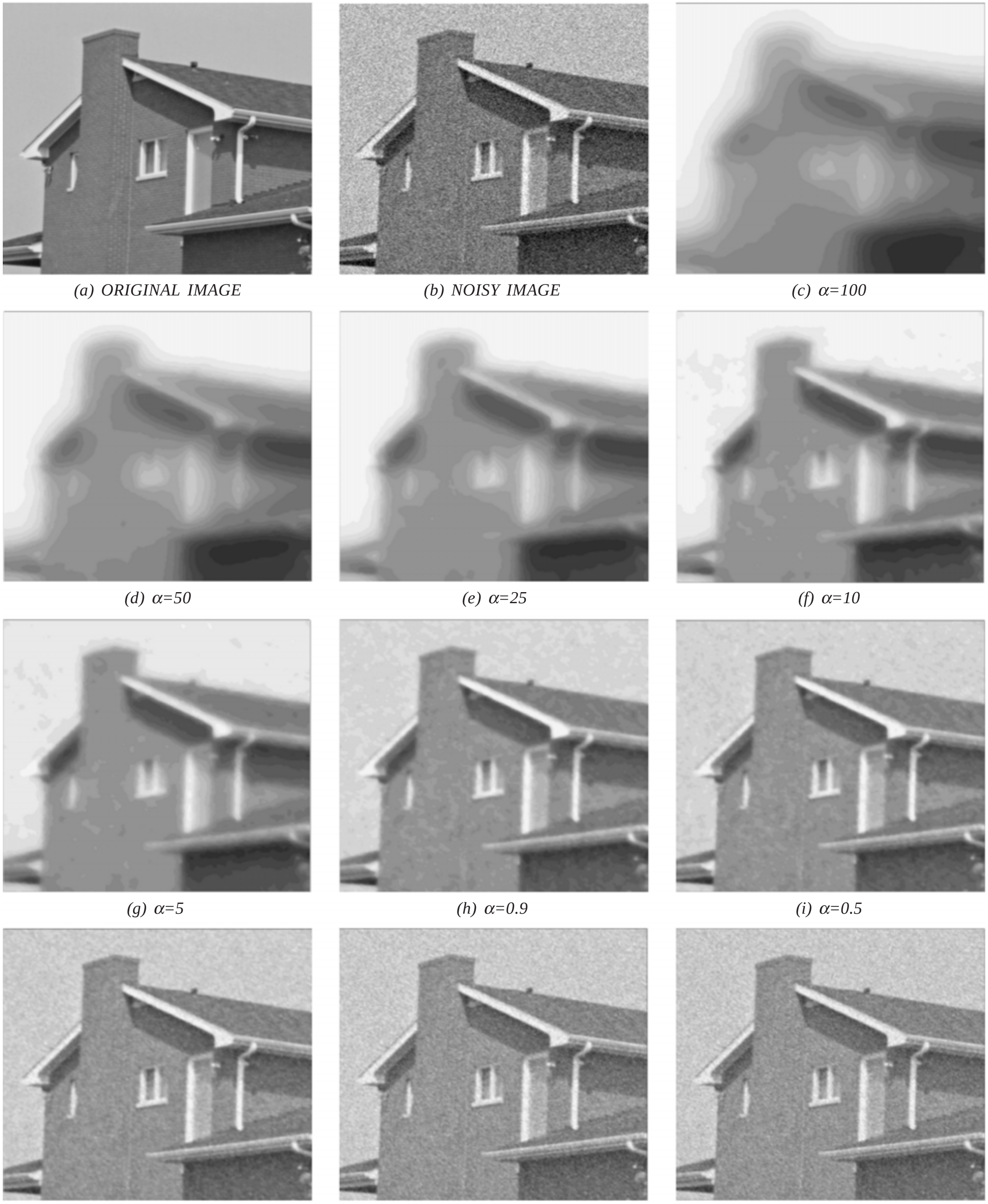


FIG. 2. AVERAGE AASD RESULTS FOR DIFFERENT CHOICES OF SELECTED PARAMETER α



(a) ORIGINAL IMAGE (b) NOISY IMAGE (c) α=100

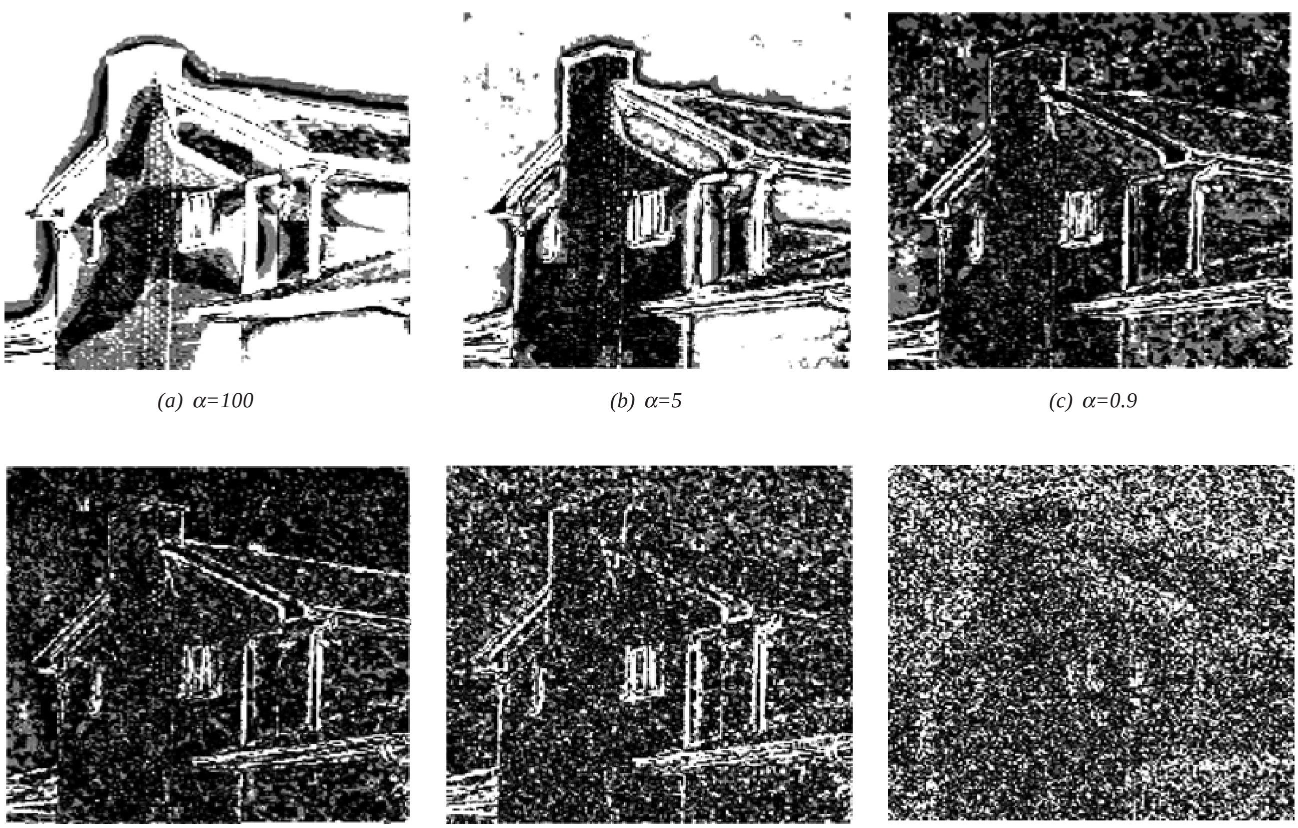
(d) α=50 (e) α=25 (f) α=10

(g) α=5 (h) α=0.9 (i) α=0.5

(j) α=0.1 (k) α=0.01 (l) α=0.001

FIG. 3(a). ORIGINAL IMAGE HOUSE (b) NOISY IMAGE WITH NOISE ADDED WITH SD=40) AND (c-l) THE DENOISED IMAGES CORRESPONDING TO THE CHOICE OF LOCAL SMOOTHNESS PARAMETER α)

starts to increase and consequently the noisy image start to appear from diffusion procedure. The question arises that why this happens? , the answer is that with the decrease in α the level of diffusion from the regularization part of the given model becomes very small and as it approaches to zero the diffused signal approaches to the original image which is noisy and consequently the oscillations increase in the signal and noise start to appear in computed low frequency signal. One can observe from the given error images that the error in the diffusion process slightly decreases when we decrease the parameter α which is observed as darker image or the image with very low grey level at each next step. From the overall performance of the approximation model it is observed that the regularization parameters have crucial importance in the computational procedure and error reduction. As the sharp edges are observed in over all computations, this is because of the contributions from the TV regularization approach which produces the enhanced edges in the images. The Fig. 4(a-f) is given as Absolute square diffusion error associated to various choices of the smoothness parameter α. The reader can observe that the error image becomes darker (convergence) as the regularization parameter decreases, but as the value of α becomes very smaller (Less than 0.5) the error slightly rises in the image and white grains (Noise) start to appear in the error image.



(a) α=100 *(b) α=5* *(c) α=0.9*

(d) α=0.1 *(e) α=0.01* *(f) α=0.001*

FIG. 4(a-f). ABSOLUTE SQUARE DIFFUSION ERROR ASSOCIATED TO VARIOUS CHOICES OF THE SMOOTHNESS PARAMETER α

5. CONCLUSION

TV regularization denoising model (1) was successfully tested for the study of optimal choice of regularization parameters on triangular grid using FEM based implicit approximation model (7), it is observed from the experiments that the selection of triangular grid for the denoising problems is novel and suitable approach which allow the local adaptive choice of the given regularization parameters on unstructured grids. It is also observed from the computational results that these local regularization parameters play a major role in improving the computational efficiency of the given approximation model. The results of the error estimates given in Table are provided here to observe the performance of the local choice of smoothness parameters along with the efficiency of the given FEM scheme. For more details we refer the reader to read the previous section and check the numerical computations. Some a posteriori confidence measures for TV regularization are under consideration as a work under progress and will appear in forth coming papers.

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