Analytical Solutions of Viscoelastic Flow through Porous Channels

GHULAM QADIR MEMON*, MUHAMMAD ANWAR SOLANGI** AND AHSANULLAH BALOCH***

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ABSTRACT

Viscoelastic flow in channel having transient hydrodynamic behavior, filled with and without porous medium is addressed. The boundary value problem is investigated through analytical and numerical solutions, for the governing system of partial differential equations, arising in the study for flow of viscoelastic fluids. Analytical solutions in terms of velocity, normal stress and shear stress at different values of time, viscosity and Darcy's number are obtained for constant viscosity Oldroyd-B constitutive model. Lie group technique is adopted to find solutions through symmetry of differential equations, whilst numerical solutions are realized by employing ND Solve, Mathematica Solver. Lie group technique is compared against numerical solutions by employing ND Solve, Mathematica Solver. The analytical solutions are observed in good agreement with the numerical solutions.

Key Words: **Lie Group Method, Viscoelastic Flow, Porous Media, Exact Solution.**

1. INTRODUCTION

Viscoelastic fluid flow through porous channels is termed as of practical interest in many investigations, which during deformation illustrate the mixture of both viscous and elastic components [1-5]. Such materials may include toothpaste, paint, blood, oil, cookie dough, soap solutions, cosmetic and etc.

Viscoelastic effects includes shear-thinning and thickening, strain-softening and hardening, viscoelastic stresses (normal and shear) and time-dependent rheological phenomena. The governing equation to model the flow of viscoelastic fluids through porous media, recently developed by Tan, et. al. [6-9], adapts to Darcy-Brinkman model.

The investigation is addressed in terms of analytical and numerical solutions of a boundary value problem for governing system of partial differential equations arising in the study of the flow of viscoelastic fluids through non porous and porous medium obeying the constant viscosity Oldroyd-B constitutive model. The analytical solutions are obtained by applying Lie group techniques, while numerical predictions are made by ND- Solve determined by Mathematica [10-11].

Symmetry group analysis based on the transformation groups known as Lie groups is most important solution technique for solving the differential equations and symmetries can be found to simplify the problem. Lie group

* Assistant Professor, Department of Mathematics, Shah Abdul Latif University, Khairpur Mirs.
** Assistant Professor, Department of Basic Sciences & Related Studies, Mehran University of Engineering & Technology, Jamshoro.
*** Professor, Department of Basic Sciences & Related Studies, Mehran University of Engineering & Technology, Jamshoro.

approach is widely applied solving the PDEs, in various fields of applied mathematics, mechanics and engineering science.

Symmetries of a differential equations form a one-parameter group of transformations in which parameter is small and guide to the reduction of the number of independent variables, has been initiated and developed in [12]. A theory presented in [13] guide to the developments in the Lie group method over previous methods. Some important studies dealing with development of the Lie group theory is made by a number of researchers [14-26].

2. PROBLEM FORMULATION

Consider the incompressible laminar flow of viscoelastic fluid in a channel filled with porous medium. The system of governing equations of flow comprises of the conservation of mass and conservation of momentum transport coupled with the Oldroyd-B constitutive model. The flow of viscoelastic fluids through porous media is assumed to be isotropic and homogeneous. The momentum equation can be modelled by using Darcy-Brinkman model and in the absence of body force, equations of continuity and momentum may be written in the following form:

$$\Delta . u = 0 \tag{1}$$

$$\frac{\rho}{\varepsilon}\frac{\partial u}{\partial t} = \frac{1}{\varepsilon}\nabla([2\mu_2 \underline{\underline{d}}] + \tau) - \nabla p - \rho u . \nabla u - \frac{\mu}{K}u \tag{2}$$

The Oldroyd-B constitutive equation describes the viscoelastic stresses in the flow can be expressed as below:

$$\lambda\frac{\partial \tau}{\partial t} = [2\mu_1 \underline{\underline{d}}] - \tau - \lambda\{u . \nabla\tau - \nabla u . \tau - (\nabla u)^T . \tau\} \tag{3}$$

In the above equations, u is the velocity vector field of flow, τ is the extra stress tensor, $\underline{\underline{d}}$ is the rate-of-strain tensor, Δ is the spatial differential operator, p is the isotropic fluid pressure (per unit density) and t is the time. The μ_1 and μ_2 are respectively the viscoelastic solute and Newtonian solvent viscosities, fluid density is denoted by ρ, whereas λ is the relaxation time of the viscoelastic fluid and K is the intrinsic permeability of the porous medium. Total viscosity μ of the viscoelastic flow is $\mu=\mu_1+\mu_2$ and is taken constant. The acceleration coefficient tensor (ε) in Equation (2) is assumed as porosity of porous media.

The equations are derived which govern the unsteady unidirectional flow of viscoelastic fluid through porous media adopting Oldroyd-B constitutive model. The derivation of such equations by employing the momentum transport equation of viscoelastic fluid and Oldroyd-B constitutive equations assuming constant pressure gradient and may be expressed in the absence of body force as follows:

$$\left.\begin{aligned} &\frac{\rho}{\varepsilon}\frac{\partial v}{\partial t} = \frac{\mu_2}{\varepsilon}\frac{\partial^2 v}{\partial y^2} + \frac{1}{\varepsilon}\frac{\partial \tau_{12}}{\partial y} - \frac{\partial p}{\partial x} - \frac{\mu}{K}v \\ &\lambda\frac{\partial \tau_{11}}{\partial t} = 2\lambda\tau_{12}\frac{\partial v}{\partial y} - \tau_{11} \\ &\lambda\frac{\partial \tau_{12}}{\partial t} = \mu_1\frac{\partial v}{\partial y} - \tau_{12} \end{aligned}\right\} \tag{4}$$

Where v(y,t) is the velocity component in axial direction and τ_{11}(y,t), τ_{12}(y,t) and τ_{22}(y,t) are the stress tensor components in axial, shear and transversal direction. As y is in the transversal direction where second normal stress vanishes (τ_{22}= 0).

The governing system of equations is written in the dimensionless form by introducing the following non-dimensional variables:

$v = v^* V_c$, $\tau=\mu V c\tau^*/L$, $y = y^*L$, $K=K^*$ and $t=t^*L/Vc$, along with material parameters: $\lambda=\lambda^*L/Vc$, $\mu_1= \mu \mu_1^*$, $\mu_2 = \mu \mu_2^*$, $Re= \rho LVc /\mu$.

Where v^*, τ^* and y^* are dimensionless velocity, stress tensor and transversal coordinates and t^* and K^* are the non-dimensional time and the non-dimensional modified permeability of the porous medium. Whilst, L is the characteristic length taken as half width of the channel and Vc is the characteristic velocity assumed as reference axial velocity $Vc=\varepsilon L^2(-\partial p/\partial x)/\mu$, then after dropping asterisk from variables for brevity, the non-dimensional equations become:

$$\left.\begin{aligned} \mathrm{Re}\frac{\partial v}{\partial t} &= 1+\mu_2\frac{\partial^2 v}{\partial y^2}+\frac{\partial \tau_{12}}{\partial y}-\frac{1}{Da}v \\ We\frac{\partial \tau_{11}}{\partial t} &= 2We\,\tau_{12}\frac{\partial v}{\partial y}-\tau_{11} \\ We\frac{\partial \tau_{12}}{\partial t} &= \mu_1\frac{\partial v}{\partial y}-\tau_{12} \end{aligned}\right\} \qquad (5)$$

Where the dimensionless Re (Reynolds Number), We (Weissenberg Number) and Da (Darcy's Number) are defined as $Re=\rho LVc /\mu$, $We=\lambda^*=\lambda Vc /L$ and $Da=K/\varepsilon L^2$ respectively.

To complete the well posed problem specification, it is necessary to prescribe initial and boundary conditions. Here initial conditions are taken from rest i.e.

$v(0,y) = 0$ when $y>0$ (6)

and boundary conditions are taken as:

$v(t,-1) = 0$ and $v(t,1) = 0$, when $t>0$ (7)

3. SOLUTION OF VISCOELASTIC FLOW THROUGH CHANNEL WITHOUT POROUS MEDIA

As Da approaches to infinity, the last Darcy's term vanishes, then the system Equation (5) written as:

$$\left.\begin{aligned} \mathrm{Re}\frac{\partial v}{\partial t} &= 1+\mu_2\frac{\partial^2 v}{\partial y^2}+\frac{\partial \tau_{12}}{\partial y} \\ We\frac{\partial \tau_{11}}{\partial t} &= 2We\,\tau_{12}\frac{\partial v}{\partial y}-\tau_{11} \\ We\frac{\partial \tau_{12}}{\partial t} &= \mu_1\frac{\partial v}{\partial y}-\tau_{12} \end{aligned}\right\} \qquad (8)$$

Subject to same initial and boundary conditions as referred in Equations (6-7).

3.1 Symmetry Analysis

Once symmetry Lie algebra of the differential equation is known, it can be used in the investigation of transformations that will reduce the equation to simpler form and it is powerful method in obtaining analytical solutions of differential equations. In this section, symmetry conditions and method for finding the Lie point symmetries are introduced.

The Operator:

$$\left.\begin{aligned} X &= \phi(t,y,v,\tau_{11},\tau_{12})\frac{\partial}{\partial t}+\xi(t,y,v,\tau_{11},\tau_{12})\frac{\partial}{\partial y} \\ &+\eta^1(t,y,v,\tau_{11},\tau_{12})\frac{\partial}{\partial v}+\eta^2(t,y,v,\tau_{11},\tau_{12})\frac{\partial}{\partial \tau_{11}} \\ &+\eta^3(t,y,v,\tau_{11},\tau_{12})\frac{\partial}{\partial \tau_{12}} \end{aligned}\right\} \qquad (9)$$

is the Lie point symmetry generator for the system of Equation (8) if:

$$\left.\begin{aligned} & X^{[2]}(1+\mu_2\, v_{yy}+\tau_{12\,y}-\mathrm{Re}\, v_t)\Big|_{(3.1)}=0 \\ & X^{[1]}(2We\,\tau_{12}\,v_y-\tau_{11}-We\,\tau_{11\,t})\Big|_{(3.1)}=0 \\ & X^{[1]}(\mu_1\, v_y-\tau_{12}-We\,\tau_{12\,t})\Big|_{(3.1)}=0 \end{aligned}\right\} \qquad (10)$$

Where first and second extended infinitesimal generator of X are:

$$X^{[1]} = X + \eta_t^{1[1]} \frac{\partial}{\partial v_t} + \eta_y^{1[1]} \frac{\partial}{\partial v_y} + \eta_t^{2[1]} \frac{\partial}{\partial \tau_{11\,t}} + \eta_y^{2[1]} \frac{\partial}{\partial \tau_{11\,y}} + \eta_t^{3[1]} \frac{\partial}{\partial \tau_{12\,t}} + \eta_y^{3[1]} \frac{\partial}{\partial \tau_{12\,y}} \qquad (11)$$

$$X^{[2]} = X^{[1]} + \eta_{yy}^{1[2]} \frac{\partial}{\partial v_{yy}} \qquad (12)$$

In which

$$\begin{aligned}
\eta_t^{1[1]} &= D_t \eta^1 - v_t\, D_t \phi - v_y\, D_t \xi; \\
\eta_y^{1[1]} &= D_y \eta^1 - v_t\, D_y \phi - v_y\, D_y \xi \\
\eta_t^{2[1]} &= D_t \eta^2 - \tau_{11t}\, D_t \phi - \tau_{11y}\, D_t \xi; \\
\eta_y^{2[1]} &= D_y \eta^2 - \tau_{11t}\, D_y \phi - \tau_{11y}\, D_y \xi \\
\eta_t^{3[1]} &= D_t \eta^3 - \tau_{12t}\, D_t \phi - \tau_{12y}\, D_t \xi; \\
\eta_y^{3[1]} &= D_y \eta^3 - \tau_{12t}\, D_y \phi - \tau_{12y}\, D_y \xi \\
\eta_{yy}^{1[2]} &= D_y \eta_y^{[1]} - v_{ty}\, D_y \phi - v_{yy}\, D_y \xi
\end{aligned} \qquad (13)$$

Where Dx_i is the total derivative operator given as:

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + \tau_{11\,t} \frac{\partial}{\partial \tau_{11}} + \tau_{12\,t} \frac{\partial}{\partial \tau_{12}} + v_{tt} \frac{\partial}{\partial v_t} + \tau_{11\,tt} \frac{\partial}{\partial \tau_{11\,t}} + \tau_{12\,tt} \frac{\partial}{\partial \tau_{12\,t}} + v_{ty} \frac{\partial}{\partial v_y} + \ldots, \\
D_y &= \frac{\partial}{\partial y} + v_y \frac{\partial}{\partial v} + \tau_{11y} \frac{\partial}{\partial \tau_{11}} + \tau_{12y} \frac{\partial}{\partial \tau_{12}} + v_{yy} \frac{\partial}{\partial v_y} + \tau_{11\,yy} \frac{\partial}{\partial \tau_{11\,y}} + \tau_{12\,yy} \frac{\partial}{\partial \tau_{12\,y}} + v_{ty} \frac{\partial}{\partial v_t} + \ldots,
\end{aligned} \qquad (14)$$

In the operator X, according to Lie's theory, the unknown functions ϕ, ξ and η are taken independent of the derivatives of the primitive variables v, $\tau_{11}$and τ_{12}. The expansion of Equation (13) can be set into the symmetry condition of Equation (10). After equating these equations with the partial derivatives of v, τ_{11}, τ_{12} and their powers, the generators can be obtained after simplifying the over determined system of linear PDEs, which is described in the following form:

$$\begin{aligned}
&\phi_y = \phi_v = \phi_{\tau_{11}} = \phi_{\tau_{12}} = 0, \\
&\xi_t = \xi_v = \xi_{\tau_{11}} = \xi_{\tau_{12}} = 0, \\
&\eta^1_{\tau_{11}} = \eta^1_{\tau_{12}} = 0 \\
&\eta^2_v = 0, \eta^1_{vv} = 0, \\
&\phi_t - 2\xi_v = 0, \\
&\mathrm{Re}\ \xi_t + \eta^3_v + 2\mu_2\, \eta^1_{yv} = 0 \\
&\eta^3_{\tau_{12}} + \xi_y - \eta^1_v = 0, \\
&-\mathrm{Re}\ \eta^1_t + \eta^3_y - \mu_2\, \eta^1_{yy} - \eta^1_v + 2\xi_Y = 0 \\
&-\eta^2 + 2We\ \tau_{12}\ \eta^1_y - We\ \eta^2_t + \tau_{11}\ \eta^2_{\tau_{11}} \\
&+ \tau_{12}\ \eta^2_{\tau_{12}} - \tau_{11}\ \phi_t = 0 \\
&2We\ \eta^3 + 2We\ \tau_{12}\ \eta^1_v - 2We\ \tau_{12}\ \eta^2_{\tau_{11}} - \mu_1\ \eta^2_{\tau_{12}} \\
&-2\ We\ \tau_{12}\ \xi_y - 2We\ \tau_{12}\ \phi_t = 0 \\
&\eta^1_v - \eta^3_{\tau_{12}} - \xi_y + \phi_t = 0, -\eta^3 \\
&+ \mu_1\ \eta^1_y - We\ \eta^3_t + \tau_{12}\ \eta^3_{\tau_{12}} - \tau_{12}\ \phi_t = 0 \\
&\eta^1_v - \eta^3_{\tau_{12}} - \xi_y + \phi_t = 0, \\
&-\eta^3 + \mu_1\ \eta^1_y - We\ \eta^3_t + \tau_{12}\ \eta^3_{\tau_{12}} - \tau_{12}\ \phi_t = 0
\end{aligned} \qquad (15)$$

3.1.1 Lie-point Symmetries

Solution of the linear system (15) gives rise to the values of the functionsϕ, ξ, η^1 , η^2 and η^3 are:

$$\begin{aligned}
&\phi = c_2, \xi = c_1, \eta^1 = c_3 (v - t/\mathrm{Re}) + c_4, \\
&\eta^2 = 2c_3 \tau_{11} + \beta(y)\ e^{\frac{-t}{We}} \text{ and } \eta^3 = c_3 \tau_{12}
\end{aligned} \qquad (16)$$

Where ci are arbitrary constants and β(y) is an arbitrary function of y. Thus the symmetry Lie algebra of the system of Equation (8) is four-dimensional and defined by the following generators:

$$X_1 = \frac{\partial}{\partial y}, X_2 = \frac{\partial}{\partial t},$$
$$X_3 = (v - t/\mathrm{Re})\frac{\partial}{\partial v} + 2\tau_{11}\frac{\partial}{\partial \tau_{11}} + \tau_{12}\frac{\partial}{\partial \tau_{12}},$$
$$X_4 = \frac{\partial}{\partial v} \tag{17}$$
$$\text{and } X_\infty = \beta(y)e^{\frac{-t}{We}}\frac{\partial}{\partial \tau_{11}}$$

3.2 Solutions

From given generator Equation (9), the invariant solutions corresponding to X, are obtained by solving the characteristic system:

$$\frac{dt}{\phi} = \frac{dy}{\xi} = \frac{dv}{\eta^1} = \frac{d\tau_{11}}{\eta^2} = \frac{d\tau_{12}}{\eta^3}$$

For solving the problem only those operators are used which represent meaningful physical solutions of the problem consisting with the governing Equation (8). This method is used to reduce the problem of PDEs Equation (8) to solvable form.

3.2.1 Invariant Solution Corresponding to the Operator X_3+tX_2

$$X = X_3 + tX_2 = \left(v - \frac{t}{\mathrm{Re}}\right)\frac{\partial}{\partial v} + 2\tau_{11}\frac{\partial}{\partial \tau_{11}} + \tau_{12}\frac{\partial}{\partial \tau_{12}} + t\frac{\partial}{\partial t}$$

The invariant results admitted by the operator X are given as:

$$\left.\begin{aligned} v(t,y) &= \frac{t}{\mathrm{Re}} + t\,\phi_1(y) \\ \tau_{11} &= t^2\,\phi_2(y) \\ \tau_{12} &= t\,\phi_3(y) \end{aligned}\right\} \tag{18}$$

Substituting the above values into given Equation (8) system represents ordinary differentials equations of functions ϕ_1(y), ϕ_2(y) and ϕ_3(y).

$$\left.\begin{aligned} &\mu_2\phi_1^{n}(y) + \phi_3'(y) - \frac{\mathrm{Re}}{t}\phi_1(y) = 0 \\ &2Wet\phi_3(y) + \phi_3'(y) - (t+2We)\phi_2(y) = 0 \\ &\mu_1 t\phi_1'(y) - (t+We)\phi_3(y) = 0 \end{aligned}\right\} \tag{19}$$

Where prime stands for derivatives of y, solving the above system of ODEs, the following solution is obtained:

$$\left.\begin{aligned} &\phi_1(y) = c_1 Cosh\sqrt{\beta}\,y + c_2 Sinh\sqrt{\beta}\,y \\ &\phi_2(y) = \frac{2We\,\mu_1\,\beta\,t^2}{(t+We)(t+2We)}(c_1 Sinh\sqrt{\beta}\,y + c_2 Cosh\sqrt{\beta}\,y)^2 \\ &\phi_3(y) = \frac{\mu_1\sqrt{\beta}\,t}{(t+We)}(c_1 Sinh\sqrt{\beta}\,y + c_2 Cosh\sqrt{\beta}\,y) \end{aligned}\right\} \tag{20}$$

where

$$\beta = \frac{\mathrm{Re}\,(t+We)}{t(\mu\,t + \mu_2\,We)}$$

Substituting solutions Equation (20) into Equation (18), the system of Equations (8) subject to initial and boundary conditions Equations (6-7) gives the following solutions:

$$\left.\begin{aligned} &v(t,y) = \frac{t}{\mathrm{Re}}\left(1 - \frac{Cosh\sqrt{\beta}\,y}{Cosh\sqrt{\beta}}\right) \\ &\tau_{11}(t,y) = \frac{2We\,\mu_1\,\beta\,t^4}{\mathrm{Re}^2\,(t+We)\,(t+2We)}\,\frac{Sinh^2\sqrt{\beta}\,y}{Cosh^2\sqrt{\beta}} \\ &\tau_{12}(t,y) = \frac{-\mu_1\sqrt{\beta}\,t^2}{\mathrm{Re}\,(t+We)}\,\frac{Sinh\sqrt{\beta}\,y}{Cosh\sqrt{\beta}} \end{aligned}\right\} \tag{21}$$

These solutions are plotted in Figs. 1-3 for several parameters and at different time t.

The result of velocity profile is displayed in Fig. 1 that the channel velocity v increases as time proceeds and reaches at highest value as time approaches to the value of 75 time units and velocity profile tends to steady-state. Similarly, the first normal stress component τ_{11} is shown in Fig. 2 which illustrates that the normal stress τ_{11} increases with increasing time and attain an upper limit at same time level in non-linear fashion. Whilst, the shear stress componen τ_{12} is displayed in Fig. 3 which exhibits linear trend in decreases with increase in time as it shall be. There is no further change in shear-stress as time reaches beyond the value of t=75 units.

3.2.2 Invariant Solutions Related with X_2

The invariant solution related with X_2 is given in the following functions:

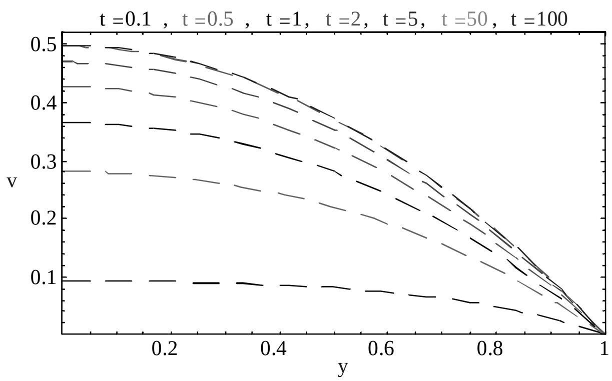

FIG. 1. ANALYTICAL SOLUTION OF THE VELOCITY v OF EQUATION (21) WITH Re=1, μ_2=8/9 AND We=1 AT DIFFERENT TIME t

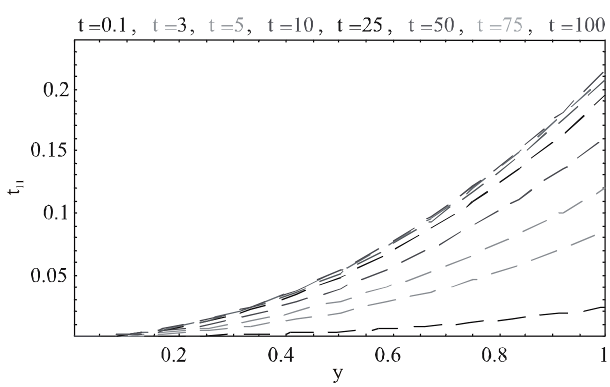

FIG. 2. ANALYTICAL SOLUTION OF THE NORMAL STRESS 11 OF EQUATION (21) WITH Re=1, μ_1=1/9, μ_2=8/9 AND We=1 AT DIFFERENT TIME t

$$\left.\begin{aligned} v(t,y) &= \psi_1(y) \\ \tau_{11}(t,y) &= \psi_2(y) \\ \tau_{11}(t,y) &= \psi_3(y) \end{aligned}\right\} \qquad (22)$$

Substituting these functions into Equations (8) yields ODEs for ψ_1(y), ψ_2(y) and ψ_3(y).

$$\left.\begin{aligned} & 1+\mu_2\,\psi_1''(y)+\psi_3'(y)=0 \\ & 2We\;\psi_3(y)\;\psi_1'(y)-\psi_2(y)=0 \\ & \mu_1\,\psi_1'(y)-\psi_3(y)=0 \end{aligned}\right\} \qquad (23)$$

Subject to boundary conditions:

$\psi_1(-1) = 0$ and $\psi_1(1) = 0$ (24)

After integration above system result in:

$$\left.\begin{aligned} \psi_1(y) &= \frac{1}{\mu}\left(c_1\,y+c_2-\frac{1}{2}y^2\right) \\ \psi_2(y) &= \frac{2We\,\mu_1}{\mu^2}(c_1-y)^2 \\ \psi_3(y) &= \frac{\mu_1}{\mu}(c_1-y) \end{aligned}\right\} \qquad (25)$$

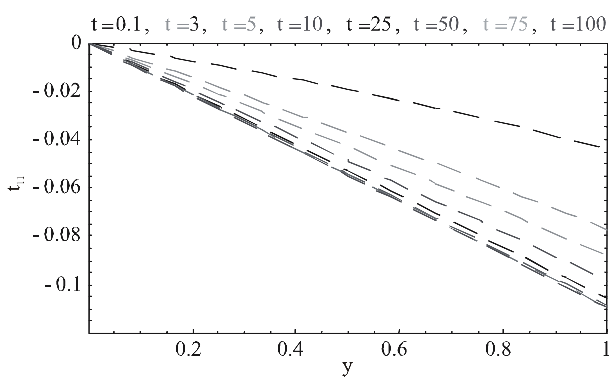

FIG. 3. ANALYTICAL SOLUTION OF THE SHEAR STRESS τ_{12} OF EQUATION (21) WITH Re=1, μ_1=1/9, μ_2=8/9 AND We=1 AT DIFFERENT TIME t

Applying the boundary conditions of Equation (24) admit the steady-state solutions

$$\left.\begin{aligned} v(y) &= \frac{1}{2\mu}(1-y^2) \\ \tau_{11}(y) &= \frac{2We\,\mu_1}{\mu^2}\,y^2 \\ \tau_{12}(y) &= \frac{-\mu_1}{\mu}\,y \end{aligned}\right\} \qquad (26)$$

Steady-state solutions are plotted in Figs. 4-6 at different values of μ. The Figs. 4-5 illustrate that if with small fluid viscosity μ, then velocity profile and normal stress component τ_{11} has large values and if channels having large viscosity μ, then velocity profile and normal stress component τ_{11} decreases and has small values and the Fig.6 show that in the steady-state, if channels having small viscosity μ, then shear stress component τ_{12} has small values and if with large viscosity μ, then shear stress component τ_{12} increases and also has large values.

3.3 Numerical Solutions

For the system of Equation (8) subject to initial and boundary conditions Equations (6-7), numerical solution is resolute adopting function NDSolve of Mathematica solver. Solutions are plotted in the Figs. 7-9, with increasing time to compare against above analytical solution obtained by Lie Group technique and displayed in Figs. 1-3. Fig. 7 illustrates that as time proceeds from rest, the velocity profile of the flow increase in parabolic fashion and reached at maximum value of v=0.5 from transition to steady-state. Similarly, in Fig. 8 the normal stress component τ_{11} has also similar trend of increase in non-linear style as time increased from initial state and reached at a maximum value of τ_{11}=0.22. Whilst, in Fig. 9 the shear stress component τ_{12} demonstrate linear tendency of increase in negative direction and attain at the value of τ_{12}=-0.11. The numerical results have very close agreement with analytical solutions.

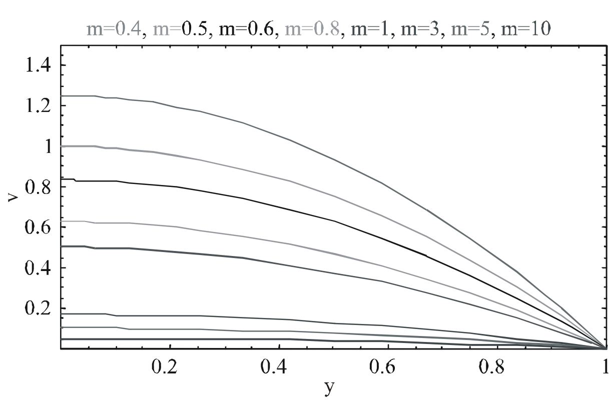

FIG. 4. STEADY-STATE SOLUTION OF VELOCITY V OF EQUATION (26) AT DIFFERENT VALUES OF μ.

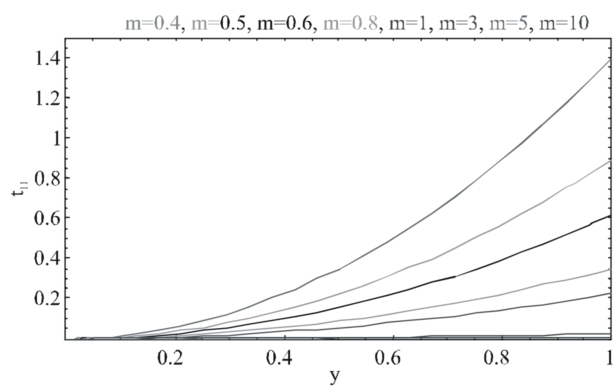

FIG. 5. STEADY-STATE SOLUTION OF THE NORMAL STRESS COMPONENT τ_{11} OF EQUATION (26) AT DIFFERENT VALUES OF μ

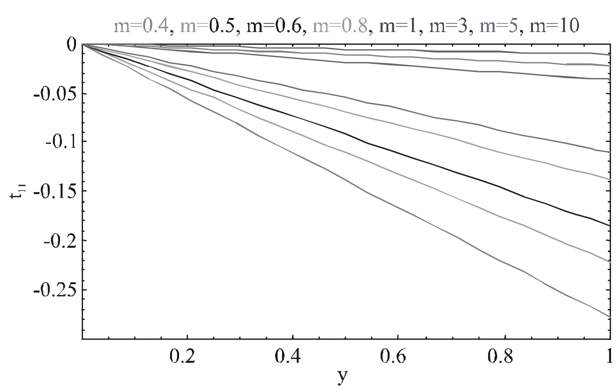

FIG. 6. STEADY-STATE SOLUTION OF THE SHEAR STRESS COMPONENT τ_{12} OF EQUATION (26) AT DIFFERENT VALUES OF μ

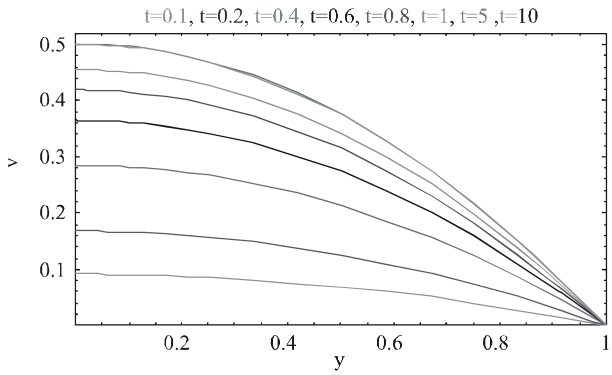

FIG. 7. NUMERICAL SOLUTION OF THE VELOCITY V OF THE SYSTEM OF EQUATION (8) SUBJECT TO INITIAL AND BOUNDARY CONDITIONS OF EQUATIONS (6-7) WITH Re=1, μ_1=1/9, μ_2=8/9 AND We=1 AT DIFFERENT TIME t

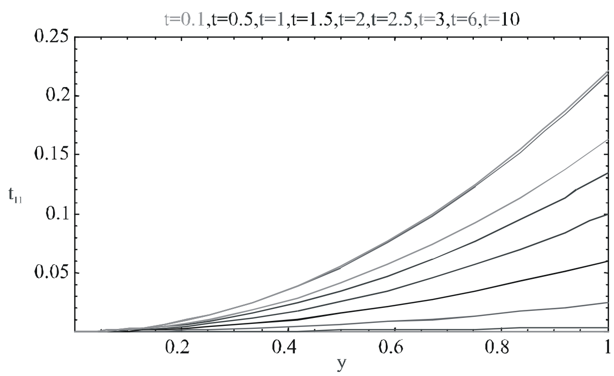

FIG. 8. NUMERICAL SOLUTION OF THE NORMAL STRESS COMPONENT τ_{11} OF THE SYSTEM OF EQUATION (8) SUBJECT TO INITIAL AND BOUNDARY CONDITIONS OF EQUATIONS (6-7) WITH Re=1, μ_1=1/9, μ_2=8/9 AND We=1 AT DIFFERENT TIME t

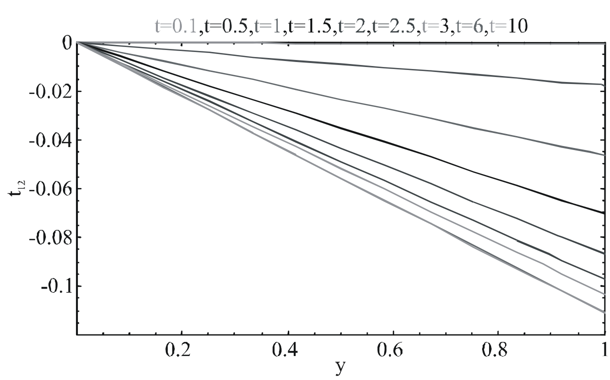

FIG. 9. NUMERICAL SOLUTION OF THE NORMAL STRESS COMPONENT τ_{12} OF THE SYSTEM OF EQUATION (8) SUBJECT TO INITIAL AND BOUNDARY CONDITIONS OF EQUATIONS (6-7) WITH Re=1,μ_1=1/9, μ_2=8/9 AND We=1 AT DIFFERENT TIME t

4. SOLUTION OF VISCOELASTIC FLOW THROUGH CHANNEL FILLED WITH POROUS MEDIA

The system of Equation (5) represents the flow of viscoelastic fluid in channels filled with porous media adopting Oldroyd-B constitutive model, solved as:

4.1. Lie-point Symmetries

The problem of Equation (5) can be admitted into Lie group transformations if and only if:

$$
\begin{aligned}
&X^{[2]}(1+\mu_2\, v_{yy}+\tau_{12y}-v/Da-\mathrm{Re}\, v_t)\Big|_{(2.5)}=0\\
&X^{[1]}(2We\,\tau_{12}\, v_y-\tau_{11}-We\,\tau_{11\,t})\Big|_{(2.5)}=0\\
&X^{[1]}(\mu_1\, v_y-\tau_{12}-We\,\tau_{12\,t})\Big|_{(2.5)}=0
\end{aligned}
\tag{27}
$$

Substituting the expansion of Equation (13) into the symmetry conditions of Equation (27). Then equating and separating them by the derivatives of v, τ_{11}, τ_{12} and their powers lead to the over determined system of linear partial differential equations and after solving the system of linear PDEs, the result of the linear equating system gives rise the values of ϕ, ξ, η^1, η^2 and η^3 in the following form:

and

$$
\begin{aligned}
&\phi=c_1,\ \xi=c_2,\ \eta^1=c_3\,(v-Da)+c_4\, e^{\frac{-t}{\mathrm{Re}\,Da}},\\
&\eta^2=2c_3\,\tau_{11}+f(y)\, e^{\frac{-t}{We}}\\
&\text{and}\\
&\eta^3=c_3\tau_{12}
\end{aligned}
\tag{28}
$$

where f(y) is an arbitrary function of y.

In Equation (28), ci are arbitrary constants and symmetry Lie algebra of system of partial differential of Equations (5) is four-dimensional and spanned by the following generators:

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial y},$$
$$X_3 = (v - Da)\frac{\partial}{\partial v} + 2\tau_{11}\frac{\partial}{\partial \tau_{11}} + \tau_{12}\frac{\partial}{\partial \tau_{12}}$$
$$X_4 = e^{\frac{-t}{\mathrm{Re}\,Da}}\frac{\partial}{\partial v} \tag{29}$$

and

$$X_\infty = f(y)\, e^{\frac{-t}{We}}\frac{\partial}{\partial \tau_{11}}$$

4.2 Solutions

Here only those operators are used that related to solution of physical problem of Equation (5) subject to initial and boundary conditions of Equations (6-7). Method of solutions depends on the applications of Lie group of transformation related with one-parameter to the system of partial differential of Equation (5).

4.2.1 Invariant Solution Corresponding to $X_1+\alpha X_4$

$$X = X_1 + \alpha X_4 = \frac{\partial}{\partial t} + \alpha\, e^{\frac{-t}{\mathrm{Re}\,Da}}\frac{\partial}{\partial v}$$

The invariant solutions under the operator $X_1+\alpha X_4$ is given by:

$$\left.\begin{aligned} v(t,y) &= \beta_1(y) - \mathrm{Re}\,Da\,\alpha\; e^{\frac{-t}{\mathrm{Re}\,Da}} \\ \tau_{11}(t,y) &= \beta_2(y) \\ \tau_{12}(t,y) &= \beta_3(y) \end{aligned}\right\} \tag{30}$$

Substituting Equation (30) into the system of Equation (5) yields system of ODEs for $\beta_1(y)$, $\beta_2(y)$ and $\beta_3(y)$.

$$\left.\begin{aligned} &\mu_2\,\beta_1''(y) + \beta_3'(y) - \frac{1}{Da}\beta_1(y) + 1 = 0 \\ &2We\,\beta_3(y)\,\beta_1'(y) - \beta_2(y) = 0 \\ &\mu_1\,\beta_1'(y) - \beta_3(y) = 0 \end{aligned}\right\} \tag{31}$$

In Equation (31), prime stands for derivatives of y. It can be seen that this system of Equation (31) admit the following solutions:

$$\left.\begin{aligned} \beta_1(y) &= c_1 \,Cosh\frac{y}{\sqrt{\mu\,Da}} + c_2\,Sinh\frac{y}{\sqrt{\mu\,Da}} + Da \\ \beta_2(y) &= \frac{2We\,\mu_1}{\mu\,Da}\left(c_1\,Sinh\frac{y}{\sqrt{\mu\,Da}} + c_2\,Cosh\frac{y}{\sqrt{\mu\,Da}}\right)^2 \\ \beta_3(y) &= \frac{\mu_1}{\sqrt{\mu\,Da}}\left(c_1\,Sinh\frac{y}{\sqrt{\mu\,Da}} + c_2\,Cosh\frac{y}{\sqrt{\mu\,Da}}\right) \end{aligned}\right\} \tag{32}$$

Substituting the values of Equation (32) into Equation (30) and applying conditions of Equations (6-7), then the system of Equation (5) admit the following solutions:

$$\left.\begin{aligned} v(t,y) &= Da\left(1 - e^{\frac{-t}{\mathrm{Re}\,Da}}\right)\left(1 - \frac{Cosh\frac{y}{\sqrt{\mu\,Da}}}{Cosh\frac{1}{\sqrt{\mu\,Da}}}\right) \\ \tau_{11}(t,y) &= \frac{2We\,\mu_1\,Da}{\mu}\left(1 - e^{\frac{-t}{\mathrm{Re}\,Da}}\right)^2 \frac{Sinh^2\frac{y}{\sqrt{\mu\,Da}}}{Cosh^2\frac{1}{\sqrt{\mu\,Da}}} \\ \tau_{12}(t,y) &= -\mu_1\sqrt{\frac{Da}{\mu}}\left(1 - e^{\frac{-t}{\mathrm{Re}\,Da}}\right)\frac{Sinh\frac{y}{\sqrt{\mu\,Da}}}{Cosh\frac{1}{\sqrt{\mu\,Da}}} \end{aligned}\right\} \tag{33}$$

These solutions of Equation (33) are expressed in Figs. 10-12 for several of parameters at different time t.

The time dependent effect on the axial velocity, normal stress component and shear stress are displayed in Figs. 10-12 respectively. Fig. 10 illustrates that the channel velocity increases as the time increases and there is an upper limit for this increase, as time t>60 velocity reaches at steady-state. Similarly effect of normal stress componen τ_{11} and shear stress component τ_{12} are shown in Figs. 11-12. These figures depict that normal stress component τ_{11} increases at an increased for values of time and there is also an upper limit for this increase, There is no further change in normal stress as time reaches beyond the value of t=60 units, and the shear stress component τ_{12} decreases as time increases and there is a lower limit for this decrease. Shear stress tends to steady-state after time t>60.

4.2.2 Invariant Solution Corresponding to X_1

The invariant solution associated with X_1 is the steady-state solution:

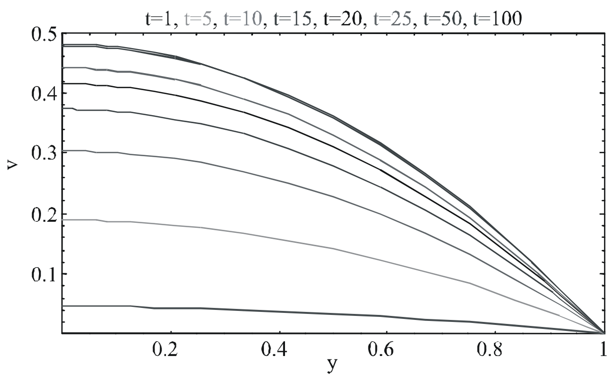

FIG. 10. ANALYTICAL SOLUTION OF THE VELOCITY V OF EQUATION (33) WITH Re=1, Da=10 AT DIFFERENT TIME t

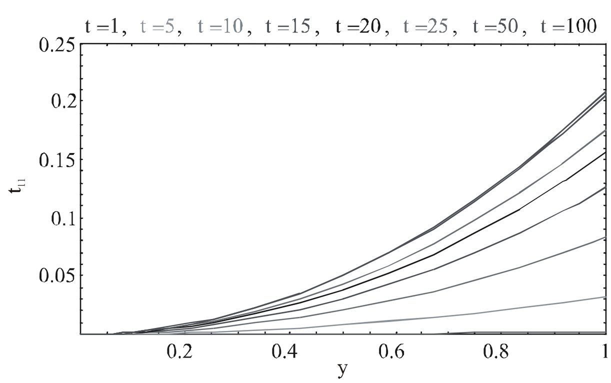

FIG. 11. ANALYTICAL SOLUTION OF THE NORMAL STRESS COMPONENT τ_{11} OF EQUATION (33) WITH Re=1, Da=10 μ_1=1/9 AND We=1 AT DIFFERENT TIME t

$$v(t,y) = \varphi(y),\quad \tau_{11} = \psi(y),\quad \text{and}\quad \tau_{12} = \phi(y) \tag{34}$$

Substituting Equation (34) into system of Equation (5) yields system of ODEs for φ(y), ψ(y), and ϕ(y):

$$\left.\begin{aligned} &1+\mu_2\,\varphi''(y)+\phi'(y)-\frac{1}{Da}\varphi(y)=0\\ &2\,We\;\phi(y)\,\varphi'(y)-\psi(y)=0\\ &\mu_1\,\varphi'(y)-\phi(y)=0 \end{aligned}\right\} \tag{35}$$

Subject to boundary conditions:

$$\varphi(-1) = 0 \text{ and } \varphi(1) = 0 \tag{36}$$

After integration of Equation (35), the result is given as:

$$\left.\begin{aligned} &\varphi(y)=d_1\,Cosh\frac{y}{\sqrt{\mu\,Da}}+d_2\,Sinh\frac{y}{\sqrt{\mu\,Da}}+Da\\ &\psi(y)=\frac{2\,We\,\mu_1}{\mu\,Da}\left(d_1\,Sinh\frac{y}{\sqrt{\mu\,Da}}+d_2\,Cosh\frac{y}{\sqrt{\mu\,Da}}\right)^2\\ &\phi(y)=\frac{\mu_1}{\sqrt{\mu\,Da}}\left(d_1\,Sinh\frac{y}{\sqrt{\mu\,Da}}+d_2\,Cosh\frac{y}{\sqrt{\mu\,Da}}\right) \end{aligned}\right\} \tag{37}$$

Applying the boundary conditions of Equation (36), system of partial differential of Equations (5) admit the steady-state solutions as under"

$$\left.\begin{aligned} &v(t,y)=\varphi(y)=Da\left(1-\frac{Cosh\frac{y}{\sqrt{\mu\,Da}}}{Cosh\frac{1}{\sqrt{\mu\,Da}}}\right)\\ &\tau_{11}(t,y)=\psi(y)=\frac{2\,We\,\mu_1\,Da}{\mu}\;\frac{Sinh^2\frac{y}{\sqrt{\mu\,Da}}}{Cosh^2\frac{1}{\sqrt{\mu\,Da}}}\\ &\tau_{12}(t,y)=\phi(y)=-\mu_1\sqrt{\frac{Da}{\mu}}\;\frac{Sinh\frac{y}{\sqrt{\mu\,Da}}}{Cosh\frac{1}{\sqrt{\mu\,Da}}} \end{aligned}\right\} \tag{38}$$

Steady-state solutions are plotted in Figs. 13-15 at different values of Da.

The steady-state velocity, normal stress component and shear stress are displayed in Figs. 13-15 respectively. The Figs. 13-14 shows that in the steady-state, if channels having small Da, then steady velocity v and steady normal stress component τ_{11} have small values that if permeability decreases, then resistance increases and hence velocity decreases and normal stress τ_{11} also decreases in the steady-state, and Fig .15 shows that in the steady-state, if channels having small Da, steady shear stress component τ_{12} have large values that is if permeability decreases, then shear stress τ_{12} increases in the steady state.

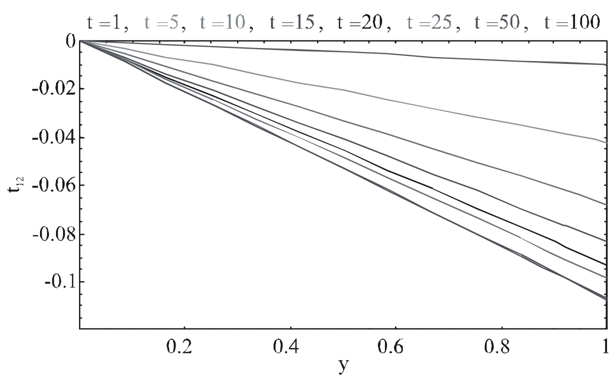

FIG. 12. ANALYTICAL SOLUTION OF THE SHEAR STRESS COMPONENT τ_{12} OF EQUATION (33) WITH Re=1, Da=10 τ_1=1/9 AND We=1 AT DIFFERENT TIME t

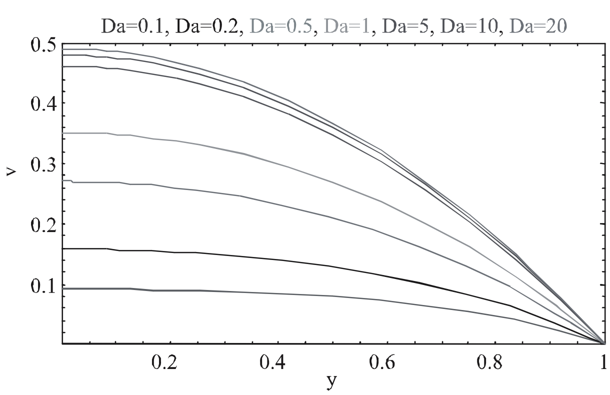

FIG. 13. STEADY-STATE SOLUTION OF THE VELOCITY V OF EQUATION (38) AT DIFFERENT VALUES OF Da

4.3 Numerical Solution

Numerical solution are obtained for the system of PDEs of Equation (5) subject to initial and boundary conditions of Equations (6-7) using NDSolve in Mathematica solver and are plotted in the Figs. 16-18 with increasing time. Fig. 16 shows that as time proceeds, the channel velocity increases and reaches at steady-state as time approaches beyond six units (t>6). Similarly, in Fig. 17 the normal stress component τ_{11} illustrate increases with respect to time and achieve steady-state at same time level with similar non-linear fashion. Whilst, in Fig. 18 the behaviour of shear stress is illustrated which clearly indicate the linear trend. All numerical results are comparable with analytical solutions.

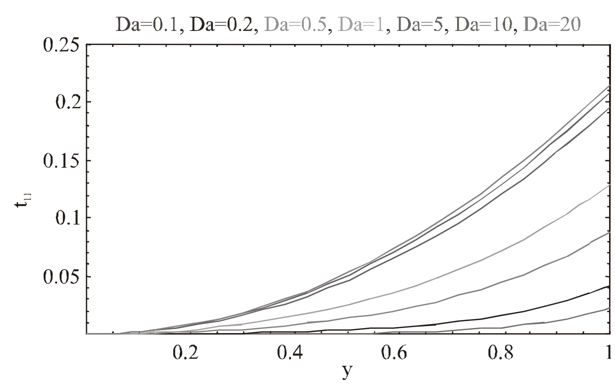

FIG. 14. STEADY-STATE SOLUTION OF THE NORMAL STRESS COMPONENT τ_{11} OF EQUATION (38) AT DIFFERENT VALUES OF Da

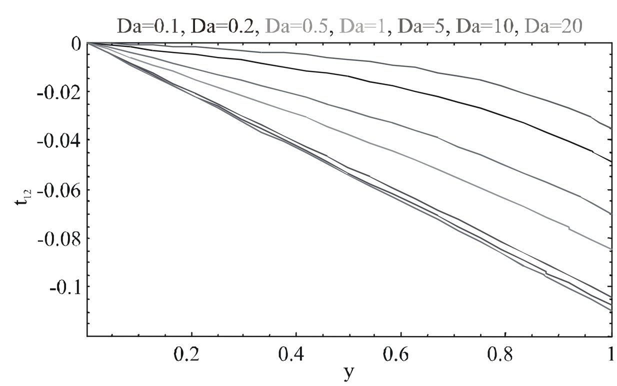

FIG. 15. STEADY-STATE SOLUTION OF THE SHEAR STRESS COMPONENT τ_{12} OF EQUATION (38) AT DIFFERENT VALUES OF Da

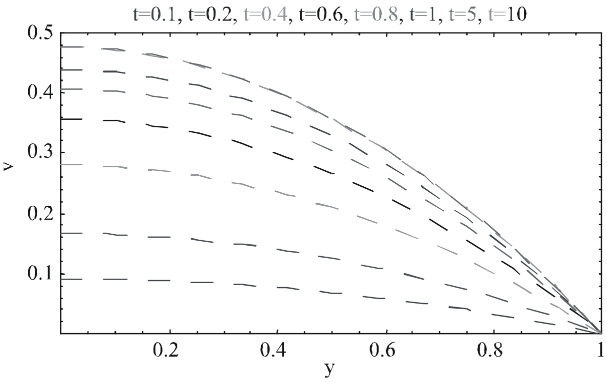

FIG. 16. NUMERICAL SOLUTION OF THE VELOCITY V OF SYSTEM OF EQUATIONS (5-7) WITH Da=10, Re=1, μ_1=1/9, μ_2=8/9 AND We=1 AT DIFFERENT TIME t

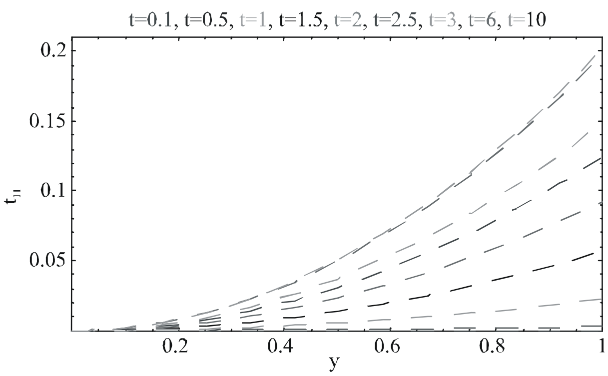

FIG. 17. NUMERICAL SOLUTION OF THE NORMAL STRESS COMPONENT τ_{11} OF THE SYSTEM OF EQUATIONS (5-7) WITH Da=10, Re=1, μ_1=1/9, μ_2=8/9 AND We=1 AT DIFFERENT TIME t

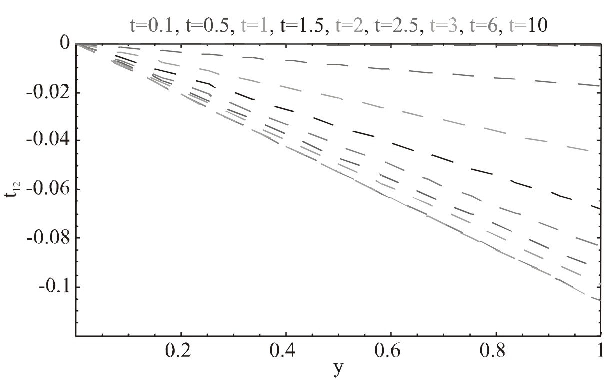

FIG. 18. NUMERICAL SOLUTION OF THE SHEAR STRESS COMPONENT τ_{12} OF THE SYSTEM OF EQUATIONS (5-7) WITH Da=10, Re=1, μ_1=1/9, μ_2=8/9 AND We=1 AT DIFFERENT TIME t

5. CONCLUSIONS

Analytical solutions are obtained successfully by employing Lie group technique for both velocity profile and non-linear Oldroyd-B stress constitutive equation coupled with momentum equation. From the results, it is observed that velocity and first normal stress components have increasing trends against increasing time and turns to be steady state at non-dimensional time greater than 50 and smaller than 75 units respectively. Whereas shear stress component decreases as time increases and becomes steady state at time greater than 75.

Whilst, in the steady state, velocity profiles and first normal stress components, at different values of viscosity and Darcy's number are observed with increasing trend against increasing values of viscosity and Darcy's number. Whereas, shear stress component is observed decreasing trend as viscosity or Da increases. Comparison has been made against analytical and numerical solutions and is found very close agreement to one another.

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