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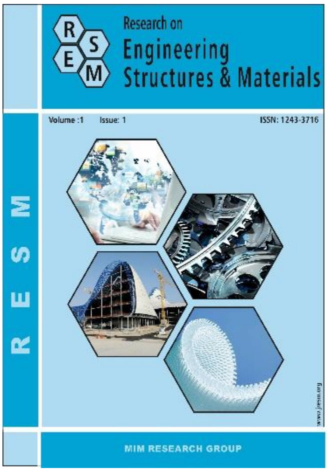

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Jouni Freund*, Alp Karakoç

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Research Article

Shear and torsion correction factors of Timoshenko beam model for generic cross sections

Jouni Freund*[1], Alp Karakoç[2]

[1]*Department of Applied Mechanics, Aalto University, Espoo, Finland*
[2]*Department of Forest Products Technology, Aalto University, Espoo, Finland*

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Abstract

A refined Timoshenko beam model which takes into account warping of cross sections is presented. The model extends St. Venant's theory of uniform torsion to a generic loading of beam. Kinetic and kinematic assumptions, virtual work expression of full elasticity problem, and principle of virtual work are used to bring the presentation to the usual context of engineering models. A new definition of the warping displacement in terms of a variational problem is one of the outcomes. Warping displacements, refined constitutive equations, and correction factors for rectangle, open annular, and angle cross sections of isotropic material are used as application examples. Shear correction factors are found to be purely geometrical quantities depending only on the shape of cross section, which contradicts many findings in literature.

1. Introduction

The standard Timoshenko beam model assumes that cross sections of beam move as rigid bodies in deformation which is a well-known source for modelling error. In refined beam models, a warping displacement part of the kinematic assumption aims to reduce the modelling error due to that source. Improved kinematics have an effect on the constitutive equation and may also affect the beam equations. Effective properties and corrections factors for the standard constitutive equation are the most popular ways to represent the effect on the constitutive equation. Warping in pure torsion is a classical topic and St. Venant's theory of uniform torsion is considered as the correct way to calculate the effective polar moment of a cross section [1]. Discussion on the warping displacement and correction factor e.g. in pure shear seems not to have settled yet and, even in the simplest case of rectangular cross section of isotropic material, a general agreement on methodology and value of the factor is lacking [2-12].

A generic computational method for finding the warping displacement and the effective constitutive equation of an elastic Timoshenko beam is suggested. The method exploits the ideas presented already in [4] but combines these in a novel manner to extend St. Venant's theory of uniform torsion to generic loading of beam. The starting point consists of virtual work expression of full elasticity problem and a set of kinematic assumptions. To end up with a coherent theory, principle of virtual work is used to find the equations associated with the standard beam displacement and warping displacement parts. Although an 'a priori' selection of the warping displacement is possible [13-14], finding the correct form seems to require the more generic approach of the present study even in simple cases. Defining the refined model by an explicit set of kinetic and kinematic assumptions brings the presentation to the usual context of engineering models and

*Corresponding author: jouni.freund@aalto.fi
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should be relatively easy to follow. Consistency with the classical theories for uniform torsion by St. Venant and thin walled elastic beam by Vlasov [15] are novelties of the present theory.

Warping displacements, refined constitutive equations, and correction factors for rectangle, open annular, and angle cross sections of isotropic material are used as application examples. Effective rigidities in torsion, obtained in the examples, coincide with St. Venant's theory of uniform torsion in all cases. However, in contrast to many findings in literature, shear correction factor turns out to be a purely geometrical quantity depending only on the shape of cross section.

2. Standard model

A short review of the standard Timoshenko beam model derivation aids in understanding the steps with the refined model. The domain of the prismatic body is denoted by $\Omega \times Z \subset \mathbb{R}^3$ in which the cross section $\Omega \subset \mathbb{R}^2$ is constant for simplicity and $Z \subset \mathbb{R}$. The standard beam model assumes that cross sections move as rigid bodies in deformation, i.e. according to kinematical assumption

$$\vec{u}_r = \vec{u}_0 + \vec{\theta}_0 \times \vec{\rho}, \tag{1}$$

in which the first and second terms on the right hand side describe displacement due to translation $\vec{u}_0$ and rotation $\vec{\theta}_0$ of the cross section, respectively. If the z-axis is aligned with the axis of the prismatic body, the relative position vector $\vec{\rho} = x\vec{i} + y\vec{j} \in \Omega$ and the kinetic assumption of the model can be written as

$$\sigma_{xx} = \sigma_{yy} = 0. \tag{2}$$

Actually, the normal stress components are assumed to be negligible compared e.g. to σ_{zz} due to bending. The role of the kinetic assumption is just to reduce the tendency for a too stiff behaviour of the standard beam model due to the rather severe kinematic assumption (Eq (1)).

Virtual work expression of the standard Timoshenko beam model

$$\delta W_r = -\int_Z \begin{Bmatrix} \delta\vec{\varepsilon} \\ \delta\vec{\kappa} \end{Bmatrix}^T \cdot \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} dz + \int_Z \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^T \cdot \begin{Bmatrix} \underline{\vec{f}} \\ \underline{\vec{m}} \end{Bmatrix} dz + \sum_{\partial Z} \begin{Bmatrix} \delta\vec{u}_0 \\ \delta\vec{\theta}_0 \end{Bmatrix}^T \cdot \begin{Bmatrix} \underline{\vec{F}} \\ \underline{\vec{M}} \end{Bmatrix} \tag{3}$$

follows from the virtual work expression of the full elasticity problem when the kinematic assumption (1) is applied there. The terms from left to right are virtual work expressions of internal forces, external volume forces, and external area forces. Denoting derivative with respect to the axial z-coordinate with prime, the strain measures in Eq (3) are defined by

$$\vec{\varepsilon} = \vec{u}_0' + \vec{k} \times \vec{\theta}_0, \tag{4}$$

$$\vec{\kappa} = \vec{\theta}_0', \tag{5}$$

and their work conjugates by

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int_\Omega \begin{Bmatrix} \vec{k} \cdot \vec{\sigma} \\ \vec{\rho} \times (\vec{k} \cdot \vec{\sigma}) \end{Bmatrix} dA = \begin{bmatrix} \vec{\vec{A}} & \vec{\vec{C}} \\ \vec{\vec{C}}_c & \vec{\vec{B}} \end{bmatrix} \cdot \begin{Bmatrix} \vec{\varepsilon} \\ \vec{\kappa} \end{Bmatrix} \tag{6}$$

in which the integral is over the cross section and $\vec{C}_c$ is the conjugate of $\vec{C}$. The first form of Eq (6) is the definition of force resultants. In the second form, the matrix depends on the geometry of cross section, selection of the coordinate system, and requires a material model satisfying the kinetic assumption (Eq (2)). ‘a priori’. The matrix representation, based on the linearly elastic material model, is the standard constitutive equation of the Timoshenko beam model.1

What remains after these steps, is just an exercise on principle of virtual work with expression in Eq (3) and the fundamental lemma of variation calculus. The outcome is the boundary value problem for the standard Timoshenko beam model containing as the unknown functions translation $\vec{u}_0(z)$ and rotation of the cross-section $\vec{\theta}_0(z)$. The great benefit of the dimension reduction method, outlined here in connection with the standard assumptions in Eqs (1) and (2), is that the steps are essentially the same irrespective of the details of the assumptions used.

3. Refined model

To take into account the possible warping of cross sections, the kinematic assumption in Eq (1) is replaced by

$$\vec{u} = \vec{u}_r + \Delta\vec{u} \tag{7}$$

in which the warping displacement $\Delta\vec{u}$ satisfies orthogonality $\vec{u}_r \perp \Delta\vec{u}$ (specified later in more detail) to make the decomposition unique. The starting point corresponds to representation e.g. in [4] and in many other references about refined beam theories. Here, however, the warping displacement is expressed in terms of the strain measures in Eqs (4) and (5) as

$$\Delta\vec{u}(x,y,z) = \Delta\vec{u}_\varepsilon(x,y)\cdot\vec{\varepsilon}(z) + \Delta\vec{u}_\kappa(x,y)\cdot\vec{\kappa}(z) \tag{8}$$

in which $\Delta\vec{u}_\varepsilon$ and $\Delta\vec{u}_\kappa$ are the warping modes. Eqs (7) and (8) compose the kinematic assumptions of the refined beam model. Additional assumption

$$\vec{\varepsilon}' = \vec{\kappa}' = 0 \tag{9}$$

is imposed in the present study to stick to the idea of St. Venant’s theory of uniform torsion. The refined model does not use any kinetic assumptions.

Considering the virtual work expression of the full linear elasticity problem as given, assumptions in Eqs (7-9) define the refined model uniquely and what remains is just manipulation with steps described in connection with the standard model. The main difference is that the virtual work expression contains also the warping modes as unknowns. The same idea has been used in [16-18] for finding the effective material properties of cellular material.

3.1. Warping mode calculation

Principle of virtual work can be applied in two steps. First, assuming that the warping modes are given so that their variations vanish, virtual work expression boils down to the standard form in Eq (3). The only difference is that stress in Eq (6) takes into account the warping displacement too. Second, assuming that the standard displacement part is given so that its variation vanishes, principle of virtual work implies the definition of warping displacement in terms of a variational problem. There, the goal is to find $\Delta\vec{u} \in U(\Omega)$ such that

$$\delta W = -\int_\Omega (\delta\nabla\Delta\vec{u})_c : \vec{\vec{\sigma}} dA = 0 \quad \forall \delta\Delta\vec{u} \in U(\Omega) \tag{10}$$

in which

$$U(\Omega)=\{\Delta\vec{u}:\int_{\Omega}\vec{\lambda}_r\cdot\Delta\vec{u}dA=0\ \forall\vec{\lambda}_r\in U_r(\Omega)\} \tag{11}$$

and rigid body motions of the cross section is denoted by $U_r(\Omega)$. Constraint definition in Eq (11) is the precise meaning of condition $\vec{u}_r\perp\Delta\vec{u}$ imposed on the warping displacement in Eq (7). Although not necessary, external distributed forces have been assumed to be of the same form as $\vec{u}_r$. Then virtual work of external distributed forces vanish on the warping displacement and only the term for the internal forces remains. Solution to the variational problem is unique as rigid body motion of the cross section Ω has been excluded from the warping displacement.

As a simple example, in a pure shear of rectangle cross section with $\varepsilon_y=\varepsilon_z=0$ and $\vec{\kappa}=0$ (see Fig. 1), solution to the warping displacement is given by

$$\Delta\vec{u}=\vec{k}(\frac{x}{4}-\frac{5x^3}{3L^2})\varepsilon_x\,. \tag{12}$$

The warping displacements e.g. in [13-14, 19-21] are similar but orthogonality is not satisfied. Simple exact solutions of this form are, however, quite exceptional and the variational problem needs to be solved numerically in almost all cases of practical interest. The finite element method, used in the examples to follow, is based on the virtual work expression in Eq (10), Lagrange multiplier method to enforce the orthogonality of the displacement parts, and a non-structured mesh of quadratic triangle elements.

4. Application examples

As application examples, finite element method is used to find the warping modes and refined constitutive equations for rectangular, open annular, and angle cross sections shown in Fig. 1. Material is assumed to be linearly elastic, homogeneous, and isotropic with Young's modulus E, Poisson's ratio ν, and shear modulus $G=E/(2+2\nu)$. Difference between the standard and refined constitutive equations is quantified by correction factors. Shear correction factors are also compared with the expressions in literature. Effective torsion rigidity of the refined model coincides with the prediction by St. Venant's theory of uniform torsion.

First, the six warping modes are solved by giving the value one to each component of $\vec{\varepsilon}$ and $\vec{\kappa}$ at a time the other components being zeros. After that, the representation in Eq (8) follows from linearity. Finally, integration over the cross section, as indicated by Eq (6), gives $\vec{\vec{A}}$, $\vec{\vec{B}}$, and $\vec{\vec{C}}$ of the refined constitutive equation. In all the cases of Fig. 1, the outcome can be expressed in the form

$$\vec{\vec{A}}=GA(\kappa_1\vec{i}\vec{i}+\kappa_2\vec{j}\vec{j})+EA\vec{k}\vec{k}\,, \tag{13}$$

$$\vec{\vec{B}}=E(I_x\vec{i}\vec{i}+I_y\vec{j}\vec{j})+\kappa_3G(I_x+I_y)\vec{k}\vec{k}\,, \tag{14}$$

$$\vec{\vec{C}}=\kappa_4G\sqrt{A(I_x+I_y)}\vec{i}\vec{k}\,, \tag{15}$$

in which A, I_x and I_y are the integrals of 1, y^2 and x^2 over the cross section in the same order. The values of the four correction factors κ_1, κ_2, κ_3, and κ_4 depend on the shape of the cross section but not on the material properties. In the standard constitutive equation, $\kappa_1=\kappa_2=\kappa_3=1$ and $\kappa_4=0$.

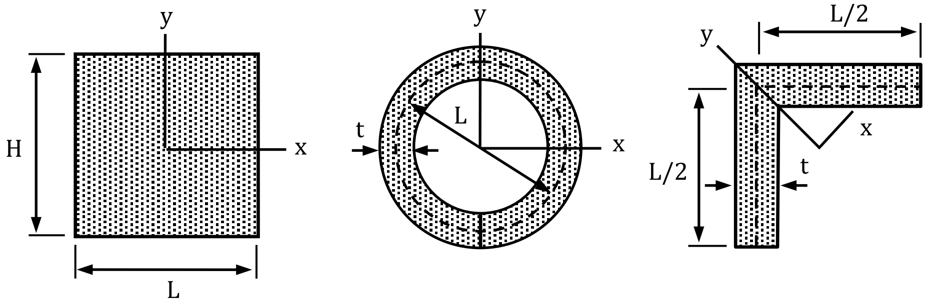

Fig. 1 Cross sections depending on geometrical parameters H, L, and t. Origin of the coordinate system is placed at the area centroid

4.1. Rectangular cross section

Fig. 2 shows the warping modes for $H/L=1$ (square) and $\nu=0.3$. The S-shaped shear warping modes for ε_x and ε_y are of the polynomial form in Eq (12). The planar warping modes for ε_z, κ_x, and κ_y are due to the Poisson effect and they vanish for $\nu=0$.

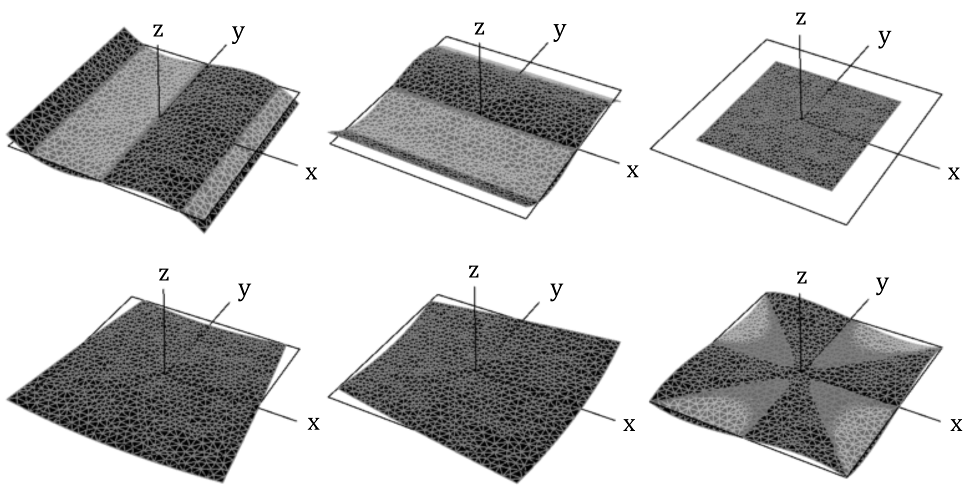

Fig. 2 Warping modes for square cross section. On the first and seconds rows, warping modes for ε_x, ε_y, ε_z and κ_x, κ_y, κ_z, respectively

Table 1 shows the correction factors as functions of shape $\alpha=H/L\in[0,1]$. It is noteworthy that shear correction factors are constants $\kappa_1=\kappa_2=5/6$. Rectangular cross section has been discussed in various references with different outcomes. Expressions of κ_1 and κ_2 in [7, 9-10, 22] depend on the Poisson's ratio and shape of the cross section. Expression in [4] depends on the Poisson's ratio but coincides with the prediction here for $\nu=0$. According to [5], the present value $5/6$ has been suggested already in 1897 by Föppl. In this particular case, torsion and area centroids coincide and the effective polar moment J by St. Venant's theory of uniform torsion and the torsion correction factor κ_3 are related by $J=\kappa_3(I_x+I_y)$.

Table 1 Correction factors as functions of shape $\alpha = H/L$

Shape (α)	Shear (κ_1)	Shear (κ_2)	Torsion (κ_3)	Connection (κ_4)
1	0.833	0.833	0.843	0.000
3/4	0.833	0.833	0.779	0.000
1/2	0.833	0.833	0.549	0.000
1/4	0.833	0.833	0.198	0.000
1/16	0.833	0.833	0.015	0.000
1/32	0.833	0.833	0.003	0.000

4.2. Open annular cross section

Open annular cross section in Fig. 1 is another benchmark case due to geometrical simplicity and poor accuracy of the standard model in a torsion problem. For a closed cross section, the standard constitutive equation is acceptable. However, if the cross section is cut to the centre point to make it open, modelling error in the standard constitutive equation is significant. Furthermore, shear and torsion modes are connected in the refined constitutive equation, whereas the connection does not exist in the standard constitutive equation. Severe warping in ε_x and κ_z modes is obvious from Fig. 3 for $t/L = 1/4$ and $\nu = 0.3$. Again, the planar modes for ε_z, κ_x and κ_y are due to the Poisson effect.

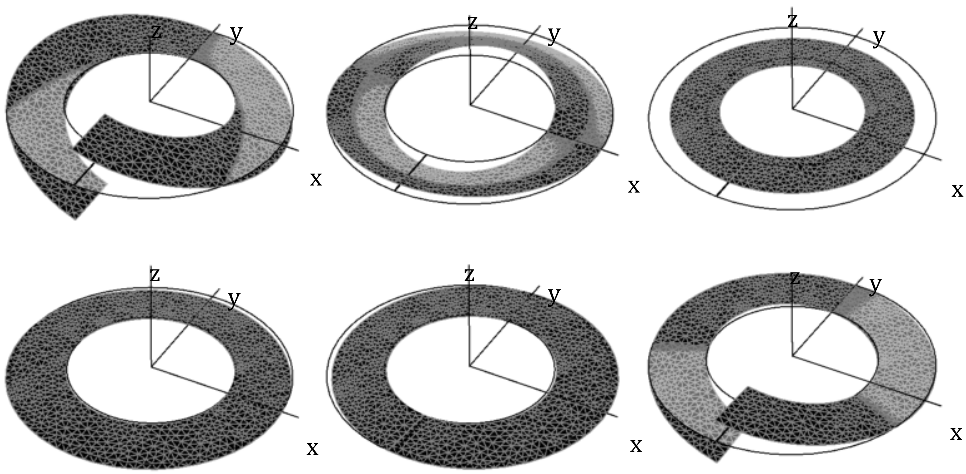

Fig. 3 Warping modes for open annular cross section. On the first and seconds rows, warping modes for ε_x, ε_y, ε_z and κ_x, κ_y, κ_z, respectively

Table 2 shows the correction factors as functions of shape $\alpha = t/L \in [0,1]$. The expressions for the solid cross section in [4, 7, 20-22] depend on the Poisson's ratio in different ways, but all coincide with the present value $\kappa_2 = 6/7$ when $\nu = 0$. The expressions of κ_2 in [4, 21] for a thin wall depend on the Poisson's ratio but agree quite well with the prediction here when $\nu = 0$. Values $\kappa_1 = 1/6$ and $\kappa_2 = 1/2$ in [2] for a thin profile are the same as obtained here. The more generic expression in [4, 22] depend on the Poisson's ratio and shape. When $\nu = 0$ the expressions coincide and give a good fit to κ_2 in Table 2. As the torsion and area centroids of an open annular cross section do not coincide, polar moment J by St. Venant's theory and torsion correction factor κ_3 cannot be compared directly but relationship $GJ = B_{zz} - C_{xz}^2 / A_{xx}$ has to be used instead.

Table 2 Correction factors as functions of shape $\alpha = t/L$

Shape (α)	Shear (κ_1)	Shear (κ_2)	Torsion (κ_3)	Connection (κ_4)
1	0.451	0.857	0.750	-0.294
3/4	0.328	0.812	0.741	-0.325
1/2	0.238	0.682	0.722	-0.338
1/4	0.184	0.551	0.688	-0.336
1/16	0.168	0.503	0.668	-0.336
1/32	0.167	0.501	0.667	-0.333

4.3. Angle cross section

The modelling error in the standard constitutive equation is significant for the thin walled angle cross sections in Fig. 1. The shear and torsion modes are connected in the refined constitutive equation unless $t/L = 1$, whereas the connection does not exist in the standard constitutive equation. Fig. 4 shows the warping modes for $\alpha = 1/4$ and $\nu = 0.3$. The out-of-plane warping modes are also in this case ε_x, ε_y, and κ_z. Although not obvious from the figure, ε_z, κ_x, and κ_y are planar warping modes due to the Poisson effect.

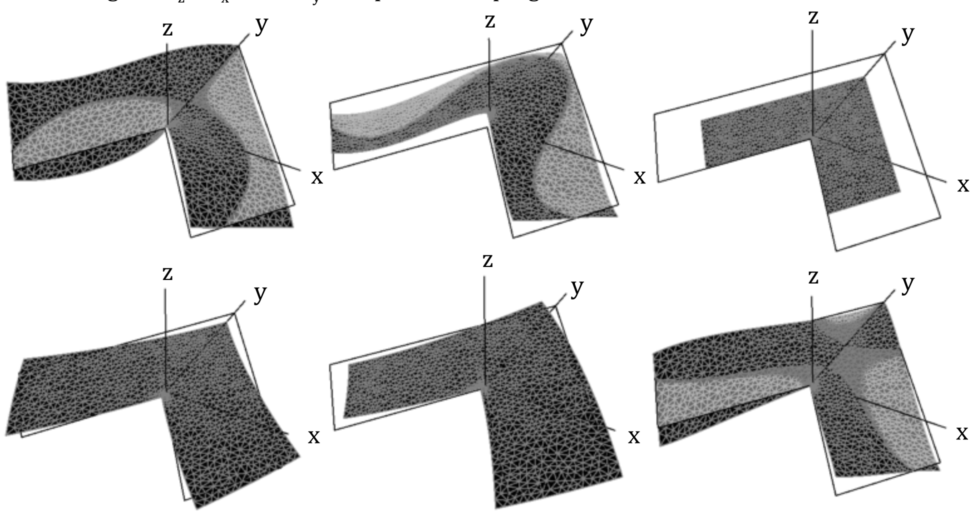

Fig. 4 Warping modes for angle cross section. On the first and seconds rows, warping modes for ε_x, ε_y, ε_z and κ_x, κ_y, κ_z, respectively

Table 3 shows the correction factors as functions of shape $\alpha = t/L \in [0,1]$. The limit case $\alpha = t/L = 1$ corresponds to a solid rectangle discussed in the rectangle cross section example with a coordinate system rotated 45°. Comparison of the first row values in Table 1 and Table 3 indicates that orientation of the axes does not affect the correction factors of a square. Angle cross section has been discussed only in a few references although it is quite common in engineering work. The shear correction factors in [12], for a relatively thin profile and $\nu = 0.3$, are in good agreement with the values in Table 3.

Table 3 Correction factors as functions of shape $\alpha = t / L$

Shape (α)	Shear (κ_1)	Shear (κ_2)	Torsion (κ_3)	Connection (κ_4)
1	0.833	0.833	0.843	0.000
3/4	0.842	0.829	0.856	0.011
1/2	0.783	0.813	0.733	-0.072
1/4	0.572	0.671	0.445	-0.282
1/16	0.443	0.440	0.276	-0.334
1/32	0.429	0.423	0.260	-0.330

5. Concluding remarks

Dimension reduction, based on the principle of virtual work and a set of kinematical and kinetic assumptions, is the common framework for the derivation of engineering models like beam, plate and shell. The consistent method has freedom only in the assumptions used and therefore any improvement is necessarily related with these. In the application here, the kinematic assumption of the standard Timoshenko beam model was modified by a warping displacement part and the kinetic assumption of the standard Timoshenko beam model was omitted. The warping modes were treated as unknown functions to be solved from a variational problem implied by the principle of virtual work. Additional kinematic assumptions in Eq (9) was used to extend St. Venant's theory of uniform torsion to generic loading of a beam.

Quoting [7] "It is perhaps surprising that even the macroscopic behaviour of a beam with a rectangular section made from homogeneous, isotropic, linearly-elastic material is not generally understood". The more recent references indicate that the problem still persist. The various non-matching refinements of the beam model are the obvious source for the non-matching results for the shear correction factor and discussions are likely to continue until some agreement is achieved. The present definition, based on set of assumptions, allows discussion of the refined model without e.g. the many technical details of solving the warping displacement numerically. Orthogonality of the displacement parts in Eq (7) is essential for a unique solution. Representation in Eq (8) preserves the idea of having translation and rotation of the cross sections as the unknowns of the beam model. Finally, assumption in Eq (9) is an extension of the assumption by St. Venant in the theory of uniform torsion. With assumption in Eq (9), the standard and refined beam models differ only in the constitutive equations.

It is noteworthy, that near warping constraints like clamped edges, assumption in Eq (9) induces modelling error that cannot be treated in a coherent manner by correction factors only. If assumption in Eq (9) is omitted, the outcome is a higher order theory for non-uniform warping discussed for thin walled cross sections in [13].

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