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On a Class of Statistical Distance Measures for Sales Distribution: Theory, Simulation and Calibration

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Abstract *While firm-level and micro issue analysis become an important part in research of international trade, only a few work is concerned about the goodness-of-fit for size distribution of firms. In this paper, we revisit the statistical aspects of firm productivity and sales revenue, in order to compare different definitions of statistical distances. We first deduce the exact form of size distribution of firms by only implementing the assumptions of productivity and demand function, and then introduce the famous g-divergence as well as its statistical implications. We also do the simulation and calibration so as to compare those different divergences, moreover, tests the combined assumptions. We conclude that minimizing Pearson χ^2 and Neyman χ^2 produces similar results and minimizing Kullback-Leibler divergence is likely to take the expense of other distance measures. Additionally, selection among different statistical distances is much more significant than demand functions.*

Key words Pareto distribution, log-normal distribution, demand function, statistical divergence, firm productivity, sales revenue

JEL Codes: **C13, C46, F14**

1. Introduction

Starting from the pioneering work by Melitz (2003), who developed a dynamic industry model with heterogeneity to analyze the intra-industry effects, firm-level analysis has become a mainstream research topic in international trade. To the extent that empirical implications have been of concern, trade theory has been aiming at understanding aggregate evidence on such topics as the factor content of trade and industry specialization (Bernard *et al*., 2003). From this perspective, many studies have also been done in understanding the micro issues, for example, the impact of firm size on productivity.

The distribution of firm productivity which is assumed to be randomly drawn grabs economists' attentions both in theoretical and empirical work. Among the literature, lots of studies assume Pareto (Arkolakis *et al*., 2012; Melitz, 2003; Head *et al*., 2014) or Log-Normal (Head *et al*., 2014; Bee *et al*., 2014) distribution of firm productivity. Upon the assumption of productivity, the distribution of sales can be determined by technology and interaction between firm and consumers. From a theoretical and statistical perspective, CES (Constant Elasticity of Substitution) expenditure function bridges the Pareto distributions of productivity and sales, and the Log-Normal distributions of productivity and sales (Helpman *et al*., 2004; Head *et al*., 2014; Mrazova *et al*., 2016), though the equilibrium resulting in Log-Normal firm size has not been deduced yet. There is also some more detailed work examining the validity of Pareto and Log-Normal sales distribution, for example, Stanley *et al*. (1995) used a Zipf plot to demonstrate that the upper tail of the size distribution of firms is too thin relative to the log normal rather than too fat.

Jumping from the scope of international trade, some empirical studies focus on fitting the actual sales. Cabral and Mata (2003) demonstrated the right-skewness of firm size distribution using Portuguese manufacturing data. And Brynjolfsson *et al.* (2014) studied the changes in the shape of Amazons sales distribution curve as well as its impact on consumer surplus gains.

Nevertheless, little literature is concerned about the goodness-of-fit for size distribution of firms. Mrazova *et al.* (2016) used KLD (Kullback Leibler Divergence) to measure the distances between estimated density and theoretical density, in order to test a new demand function developed by them, called CREMR. In fact, testing and measuring the statistical distance between the empirical distribution and the estimated distribution is of great importance if we want to test the combined assumptions, at least from a statistical standpoint.

Unavoidably, talking about the statistical distance involves the measure of statistical divergence, which is distinguished from the traditional Euclidean distance in metric space. In this paper, we introduce the well-known g-divergences that are applied widely in statistics and engineering. To our surprise, this has not been used too much in economics, in spite of its great properties in measuring the distances.

We start from the assumption of firm productivity, either Pareto or Log-Normal distributions that are proved to be very effective and empirically practical. By combining the assumptions of firm with different forms of demand systems, which further imply the exact forms of the probability distribution of firm sizes. Besides the theory, we also run simulations to get a sense of how those different g-divergences work, concluding that minimizing Pearson χ^2 and Neyman χ^2 produces similar results and minimizing Kullback-Leibler divergence is likely to take the expense of other distance measures. Finally, we use the data on US exports to Canada in 2015 showing that CES or LES are likely to be an appropriate representation comparing to Translog demand function. To the best of our knowledge, this paper is one the few works studying the firm size distribution from a distance angle.

2. Sales Distribution

Without assumptions on demand and technological constrains, firm characteristics can be linked by a behavior function by only assuming that characteristics of each firm are monotonically increasing on the number of firms, thus a hypothetical dataset of a continuum of firms (Mrazova *et al.*, 2016). Implementing this assumption, when the linked demand function and the distribution of productivity are specified, sales distribution can be determined statistically. In this section, we use three kinds of demand systems to extract the exact forms of sales distribution by assuming that firm productivity is either distributed Pareto or Log-Normal.

2.1. Pareto productivity

Suppose that the productivity ' is distributed Pareto with scale parameter φ_m and shape parameter α, so the cumulative probability function is $F(\varphi) = 1 - (\frac{\varphi_m}{\varphi})^\alpha$.
From the perspective of pro t-maximization, marginal cost equals marginal revenue. As a result, we can link sales r and productivity by $\partial r/\partial x = 1/\varphi(x)$, where *x* is the output. Accordingly, demand function indeed plays the role of bridging productivity and firm size.
We first give an example of how to deduce the size distribution of firms by using Constant Elasticity of Substitution demand function. Suppose the inverse CES demand function takes the form $p(x) = bx^{-1/a}$, we can utilize sales revenue to express firm productivity by the following steps,

express sales by ouputs, $r(x) = p(x)x = bx^{1-1/a}$,

express outputs by sales, $x = (r/b)^{\frac{a}{a-1}}$,

calculate the marginal revenue, $r'(x) = (1-1/a)bx^{-1/a}$

express firm productivity, $\varphi(x) = \frac{a}{b(a-1)}x^{1/a}$,

express firm productivity by sales, $\varphi(r) = \frac{a}{b(a-1)}(r/b)^{\frac{1}{a-1}}$. (1)

Then by the variable transformation, we know that the cumulative probability function of sales is $F(r) = F(\varphi(r)) = 1 - \varphi_m^\alpha(\frac{a}{b(a-1)})^{-\alpha}(\frac{r}{b})^{-\frac{\alpha}{a-1}}$ under the setting of CES demand function.
Table 1 summarizes the expression of productivity by sales as well as the sales distribution by three common classes of demand systems.

Table 1. Sales distribution with Pareto distribution of productivity

	Demand Form	Productivity	CDF of Sales
CES	$p(x) = bx^{-1/a}$	$\varphi(r) = \frac{a}{b(a-1)}(r/b)^{\frac{1}{a-1}}$	$F(r) = 1 - \varphi_m^{\alpha}(\frac{a}{b(a-1)})^{-\alpha}(\frac{r}{b})^{-\frac{\alpha}{a-1}}$
LES	$p(x) = \frac{a}{x+b}$	$\varphi(r) = \frac{b}{a}(\frac{a}{a-r})^2$	$F(r) = 1 - \varphi_m^{\alpha}(\frac{b}{a})^{-\alpha}(\frac{a-r}{a})^{2\alpha}$
Translog	$x(p) = \frac{1}{p}(a - b\log p)$	$\varphi(r) = \exp(\frac{r-a}{b})(\frac{r}{b}+1)$	$F(r) = 1 - \varphi_m^{\alpha}\exp(\frac{a\alpha - r\alpha}{b})(\frac{r}{b}+1)^{-\alpha}$

While we don't provide the exact steps for extracting the sales distributions with LES and Translog demand functions, it's not hard to justify the above table by using the algorithm developed for CES type. Moreover, though we just consider CES, LES and Translog here, in fact, CREMR, Linear and AIDS fall into the same forms of the sales distributions respectively (Mrazova *et al*., 2016).

2.2. Log-Normal productivity

This section assumes Log-Normal distribution of productivity with location parameter μ and scale parameter σ, so the CDF is $F(\varphi) = \Phi(\frac{\log\varphi - \mu}{\sigma})$

Where Φ is the cumulative probability function of a standard normal random variable. The way of extracting the sales distributions is similar to the last section. When the forms of demand functions and the expressions of firm productivity by firm size do not depend on the distribution of productivity, we only report the sales distributions in the table 2 below.

Table 2. Sales Distribution with Log-Normal distribution of productivity

	CDF of Sales
CES	$F(r) = \Phi(\frac{\log(r/b) - \mu(a-1) + (a-1)\log(\frac{a}{b(a-1)})}{(a-1)\sigma})$
LES	$F(r) = \Phi(\frac{2\log(\frac{a}{a-r}) + \log(b/a) - \mu}{\sigma})$
Translog	$F(r) = \Phi(\frac{r - a - b\mu + b\log(r/b+1)}{b\sigma})$

We express the sales distributions by using the original parameters in demand functions and distributions of productivity. However, sufficient parameters in each distribution are less than those in the table. For example, for the combination of LES with Pareto case, we can express the sales distribution by

[1]As same as those in Table 1

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$$F(r) = 1 - m(1 - \frac{r}{k})^h. \tag{2}$$

Accordingly, estimating m; k and h instead of φ_m ; a; b and α is sufficient to know the exact distribution of sales. We call it a generalization of parameters and will use the generalized ones for estimation in the next two sections.

3. Statistical divergences

Statistical divergences are used to measure the "distance" between two probability distributions. The importance of suitable measures of distance arises of the role play in the problems of inference and discrimination (Ullah, 1996). While the concepts of statistical distances are not widely applied in Economics, experts in other fields have appreciated them in other fields. For instance, different distances are used to identify the relation between texts of DNAs, planetary nebulae and unitary transformations (Pevzner, 1991; Steene and Zijlstra, 1994; Acin, 2001). In this section, we overview an important class of statistical divergences which is crucial in this paper.

3.1. *g*-Divergence

Suppose *f* is the actual (empirical) density and $\hat{f}$ is the estimated density. The g-divergence[22] is defined as

$$H_g(f, \hat{f}) = \int f(x) g(\frac{\hat{f}(x)}{f(x)}) dx, \tag{3}$$

Where g(.) is an arbitrary convex function. The *g*-divergence is characterized as a unique family of convex separable divergence that satisfies the information monotonicity property (Amari, 2009; Amari and Nagaoka, 2000). The *g*-divergence is not necessary a metric since the symmetric property is not satisfied for every g, i.e. $H_g(f, \hat{f}) \neq H_g(\hat{f}, f)$. Although considering this general form is promising, we need the analytical expression of distances in order to do parametric estimations. As a matter of fact, this class of divergences by defining different *g* functions have explicit statistical meanings. In this paper, we will mainly use the following.

3.1.1. Kullback-Leibler divergence

Define g(x) = -log(x), then the divergence becomes $\int f(x) \log(\frac{f(x)}{\hat{f}(x)}) dx$

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[2] Some papers name it as *f*-divergences, such as Nielsen and Nock (2013). But in order not to conflict with the notations of density, we use g here.

It's the famous Kullback-Leibler Divergence (Relative Entropy). Since minimizing KLD is equivalent to maximizing the Likelihood function, KLD estimators are just MLEs. By this property, KLD might be the most widely used divergence, for example, Mori *et al*. (2005) used this measure to define industrial localization and Rohde (2016) studied income inequality based on a symmetric extension of KLD.

3.1.2. Pearson and Neyman χ^2

Defining g(x) = $(x-1)^2$ and g(x) = $(1-x)^2$ gets Pearson and Neyman χ^2 estimators:

$$\int \frac{(\hat{f}(x) - f(x))^2}{f(x)} dx \quad \text{sau} \quad \int \frac{(\hat{f}(x) - f(x))^2}{\hat{f}(x)} dx.$$

These two estimators are applied to sets of categorical data to evaluate how likely it is that any observed difference between the sets arose by chance, which is known as χ^2 test.

3.1.3. Total variation distance

If we define $g(x) = \frac{1}{2}|x - 1|$, we can get the divergence $\frac{1}{2}\int |f(x) - \hat{f}(x)|dx.$ By this definition, the divergence is symmetrical, i.e. $H(f, \hat{f}) = H(\hat{f}, f).$

3.2. Other divergences

In fact, there are a lot of statistical distances that can be considered without restriction to g-divergences. As a robustness check and comparison, we will also use Kolmogorov-Smirnov Statistic and Hellinger Distance in the next two sections. Kolmogorov-Smirnov Statistic is defined as $\sup_x |F(x) - \hat{F}(x)|$, where *F* and $\hat{F}$ are the true and estimated cumulative probability functions with respect to *f* and $\hat{f}$. Furthermore, Hellinger Distance is $\frac{1}{\sqrt{2}}\sqrt{\int (f(x) - \hat{f}(x))^2 dx},$, to which square is in g-divergences class.

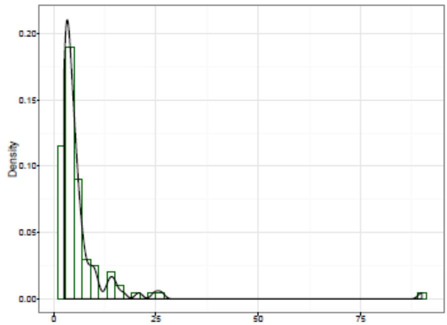
4. Simulation

Firm productivity and sales revenue are fitted by Pareto and Log-Normal distribution in many studies. Consequently, by the importance of these two probability distributions, this section presents the simulated results of Pareto and Log-Normal variables. Specifically, we simulate a set of values drawn from either
Pareto or Log-Normal distribution, then minimizing different statistical distances defined in the last section to estimate the parameters in order to get a sense of how those distances work.

4.1. Pareto simulation

In this section, we simulate Pareto variable with scale parameter 2.5 and shape parameter 1.5. Figure 1 is the histogram with density of our simulated data.

Figure 1. Pareto



In order to estimate the parameter, we minimize over each definition of statistical distance and report them for each different optimization.

Table 3. Distances between predicted density and actual density for Pareto

	KLD	Pearson χ^2	Neyman χ^2	TV	K-S	Hellinger
KLD	0.005353301	0.00027670210	0.0002773337	0.034907020	0.06827124	0.007636033
Pearson χ^2	0.005362378	0.00002096860	0.0000209194	0.006701914	**0.0698160**	0.001071308
Neyman χ^2	0.005362376	0.00002096856	0.0000209194	0.006701974	0.06981567	0.001071206
TV	0.005362377	0.00002096855	0.0000209194	0.006701856	0.06981583	0.001071267
K-S	**0.00538507**	**0.0523065900**	**0.055116100**	**0.49264730**	0.04756686	**0.10898200**
Hellinger	0.005362270	0.00002099893	0.0000209489	0.006721100	0.06979908	0.001068044

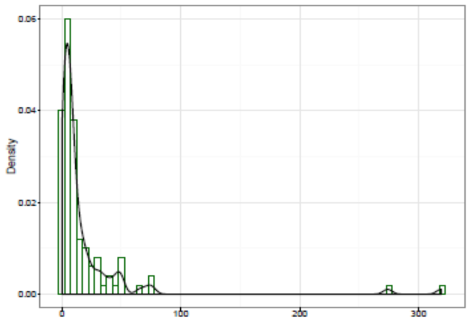
* Note that the values in the same column are rounded to the same number of digits in order to do comparisons.

The first column represents the minimized distance and each row stands for the estimated values with correspondence to the rule in the left.

4.2. Log-Normal simulation

We do a similar simulation and optimization as last section by using Log-Normal distribution with location parameter 2 and scale parameter 1.2. Below is the histogram with plotting density.

Figure 2. Log-Normal



Again by minimizing different statistical divergences, we have the following summary table.

Table 4. Distances between predicted density and actual density for Log-Normal

	KLD	Pearson χ^2	Neyman χ^2	TV	K-S	Hellinger
KLD	0.001556432	**0.019263780**	**0.017543570**	**0.1141217**	**0.0588223**	**0.02356127**
Pearson χ^2	0.002199208	0.0001307100	0.0001313402	0.01153388	0.05420171	0.002236631
Neyman χ^2	0.002199091	0.0001307102	0.0001313401	0.01153476	0.05420062	0.002236464
TV	**0.002231887**	0.0001415405	0.0001423871	0.01138590	0.05450223	0.002352665
K-S	0.002112283	0.0002154889	0.0002162137	0.01438442	0.05351154	0.002636670
Hellinger	0.002188878	0.0001318320	0.0001324514	0.01167182	0.05410512	0.002229142

* Note that the values in the same column are rounded to the same number of digits in order to do comparisons.

4.3. Discussions

Cross-comparison may be of little interest, but we can compare the values within the same distance group. We notice that minimizations of Pearson χ^2, Neyman χ^2 and total variation distance give similar estimated results with respect to all different measures and both for Pareto and Log-Normal simulations. Furthermore, such similarity is even larger for minimizing Pearson χ^2 and Neyman χ^2. Also the estimated Pearson χ^2 and Neyman χ^2 are pretty close under each rule. While it's easy to see from the forms of those two χ^2 estimators that their definitions are actually highly related and when the estimated density approximates the true density, they are closer, we do not have any powerful explanations for why TV method is also similar.
Focusing on each single goodness-of-fit, we see that the most of them have large variations among different optimization rules, when K-S statistic has the smallest

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variation3[3]. So we would expect a stable performance for K-S estimations. Moreover, KLD estimation method looks different from others, if we remove the row of KLD in the table, we can easily discover large stability of all estimators. From these perspectives, KLD and K-S distances may not be very good ones since they minimize KLD and K-S respectively by taking the expense of other distances. We can conclude that more obviously by look at the bold numbers which are the maximum of each column.

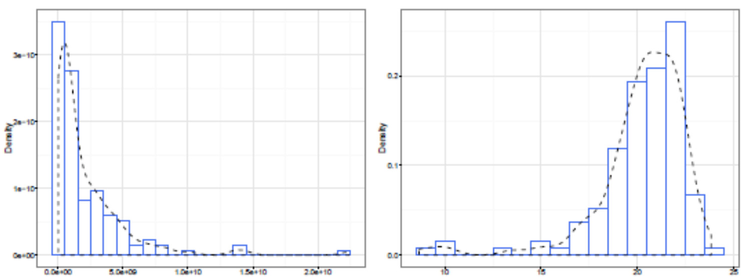
5. Calibration

In this section, we do a calibration to show how the concepts of *g*-divergences can test economics assumptions, at least from a statistical perspective. Certainly, each market can be modeled by a mixture of demand systems. And the specification of demand system associate with the assumption of firm productivity can determine the distribution of sales, which further determines the behavior of sales. However, traditional economics methods fall into regression to fit sales data, hence testing whether the assumptions hold, for example, see Tobin (1950); Bergstrom and Goodman (1973); Griliches (1958). Intuitively, smaller the statistical distances are, the more approximation of the predication, resulting in more reasonable assumptions.

We give an example of how these work. To be specific, we use aggregate manufacturer-level sales data exported from US to Canada in 2015, which can be found via US Census Bureau Economic Statistics. We will mainly focus on the performances of different demand settings and the firm productivity assumptions.

Figure 3 below are two histograms with density plot, the right one is after taking logarithm transformation.

Figure 3. Histograms



[3]The K-S column has the smallest variation

At the first glance, either Pareto or Log-Normal is likely to fit the data well. Then we turn to the *g*-divergences estimations, table 5 below reports the estimated divergences under two different assumptions of productivity and multiple demand settings. For example, the entries of the Translog column in the Pareto suitable mean the minimized values of those six divergences. And we make the minimums in each row bold.

Table 5. Distances between predicted density and actual density for selected demand functions

Pareto Productivity			
	CES	Translog	LES
KLD	0.2719110000	0.27337420	**0.1774733**
Pearson χ^2	0.0001595877	0.00015959	**0.0001596**
Neyman χ^2	**0.012463350**	0.01307877	0.04740698
TV	0.0000079794	0.00007743	**0.0000734**
K-S	**0.269321800**	0.93120610	0.37507370
Hellinger	**0.000010690**	0.00001086	0.00001104
Log-Normal Productivity			
	CES	Translog	LES
KLD	0.0289275700	0.27338700	**0.0186990**
Pearson χ^2	0.0001562284	0.00015958	**0.0001065**
Neyman χ^2	**0.000074165**	0.18178080	0.00027649
TV	**0.000013954**	0.00007979	0.00007979
K-S	**0.057254705**	0.99253730	0.15416110
Hellinger	**0.000032621**	0.00001104	0.00000533

* Note that the values in the same column are rounded to the same number of digits in order to do comparisons.

Two main assumptions are implemented in the calibration, the distribution of firm productivity and the form of demand function. Recall that, by using the combination of two assumptions from each group, we have derived the exact distribution of sales revenue in section 2. Hence, each column in the two tables represents the same form of size distribution of firms. First of all, we note that from the bold numbers, LES and Translog predict better sales distributions, in other words, result in smaller distances. In fact, none of the distances by Translog brings the smallest estimation in each row. What's more fascinating is that same demand functions are selected as the ones with the smallest distances for both Pareto and Log-Normal productivity except for total variation statistics. For instance, the assumption of LES demand function provides the best of goodness of fit for whatever the distribution of productivity. Consequently, we are confident to say that LES and CES assumptions are more likely to be in line with reality. Secondly, when it comes to the same demand system, Log-Normal productivity is always better than Pareto in all aspects of distance measures under CES, since all values in the first column below are smaller than those above. The same occurrence happens to LES except for TV. Nevertheless, when assuming Translog demand function, things are opposite other than Pearson χ^2 estimation. Finally, we note that the differences between the selections of divergence measure are certainly less significant than demand functions. Or say, the differences between columns are much less than the differences between rows, which are an evidence of the importance of choosing statistical measures. However, this is just a simple calibration example which illustrates how to compare the assumption in the market of U.S. exporting to Canada. Economics networks can be hugely complicated and cannot be fuelled explained with very simple assumptions. Also, sales data are of course not drawn from a completely random process. Thus we do not have a dinner table yet to discuss the underlying difference among distances and how to predict the sales distribution generally.

6. Conclusions

With firm-level analysis playing an essential role in international trade research, understanding micro issues such as firm productivity and firm size are not only important for theory development but also empirically useful. In this paper, we revisit the statistical perspectives of productivity and firm sales and use an efficient tool to compare different statistical divergences as well as assumptions of firm and demand functions. The simulation results show that Pearson χ^2 and Neyman χ^2 perform pretty similar and minimizing Kullback-Leibler divergence is likely to be at the expense of other distance measures. We also do a calibration on manufacturer-level data exported from US to Canada in 2015. From the empirical results, we conclude that Translog is the most impossible one to fit the demand system in the example. Moreover, we conclude that selection among different statistical distances is much more significant than different demand functions. We emphasis our work as the one of few papers studying the goodness-of-fit problem in predicting size distribution of firms, from an empirically statistical standpoint. Also, we do think more work can be done on this topic such like how to use the divergence tool to test economics assumptions.

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