GENERALIZED DUAL CONNECTIONS ON PARA-KENMOTSU MONIFOLDS

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ABSTRACT. In the context of para-Kenmotsu geometry properties of the generalized dual connections of some canonical linear connections (Levi-Civita, para-Kenmotsu canonical, Golab and Zamkovoy canonical paracontact connections) are established.

1. Introduction

Consider M a $(2n+1)$-dimensional smooth manifold, φ a tensor field of $(1,1)$-type, ξ a vector field, η a 1-form and g a pseudo-Riemannian metric on M of signature $(n+1, n)$.

DEFINITION 1.1. [**23**] *We say that* (φ, ξ, η, g) *defines an almost paracontact metric structure on M if:*

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \quad \varphi^2 = I - \eta \otimes \xi, \quad g(\varphi\cdot, \varphi\cdot) = -g + \eta \otimes \eta$$

and φ induces on the $2n$-dimensional distribution $\mathcal{D} := \ker \eta$ an almost paracomplex structure P i.e. $P^2 = I$ and the eigensubbundles $\mathcal{D}^+$, $\mathcal{D}^-$, corresponding to the eigenvalues 1, -1 of P respectively, have equal dimension n; hence $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$.

In this case, $(M, \varphi, \xi, \eta, g)$ is called *almost paracontact metric manifold*, φ *the structure endomorphism*, ξ *the characteristic vector field*, η *the paracontact form* and *g compatible metric*.

Examples of almost paracontact metric structures can be found in [**12**] and [**6**]. From the definition it follows that η is the *g*-dual of the unitary vector field ξ:

$$\eta(X) = g(X, \xi)$$

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and φ is a g-skew-symmetric operator:

$$g(\varphi X, Y) = -g(X, \varphi Y).$$

Remark that the canonical distribution $\mathcal{D}$ is φ-invariant since $\mathcal{D} = Im\varphi$ and the vector field ξ is orthogonal to $\mathcal{D}$, therefore the tangent bundle splits orthogonally:

$$TM = \mathcal{D} \oplus \langle \xi \rangle.$$

An analogue of the Kenmotsu manifold [13] in paracontact geometry will be further considered.

DEFINITION 1.2. [18] *We say that the almost paracontact metric structure* (φ, ξ, η, g) *is para-Kenmotsu if the Levi-Civita connection* ∇ *of* g *satisfies* $(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$, *for any* X, $Y \in \chi(M)$.

EXAMPLE 1.1. Let $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Set

$$\varphi := \frac{\partial}{\partial y} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \quad \xi := -\frac{\partial}{\partial z}, \quad \eta := -dz,$$

$$g := dx \otimes dx - dy \otimes dy + dz \otimes dz.$$

Then (φ, ξ, η, g) defines a para-Kenmotsu structure on $\mathbb{R}^3$.

Note that the para-Kenmotsu structure was introduced by J. Wełyczko in [22] for 3-dimensional normal almost paracontact metric structures. A similar notion called P-Kenmotsu structure appears in the paper of B. B. Sinha and K. L. Sai Prasad [20].

Properties of this structure are stated in the next Proposition.

PROPOSITION 1.1. [2] *On a para-Kenmotsu manifold* $(M, \varphi, \xi, \eta, g)$, *the following relations hold:*

$$\nabla \xi = I - \eta \otimes \xi$$

$$\eta(\nabla_X \xi) = 0,$$

$$R_\nabla(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$\eta(R_\nabla(X, Y)W) = -\eta(X)g(Y, W) + \eta(Y)g(X, W),$$

$$\nabla \eta = g - \eta \otimes \eta,$$

$$L_\xi \varphi = 0, \quad L_\xi \eta = 0, \quad L_\xi g = 2(g - \eta \otimes \eta),$$

where R_∇ *is the Riemann curvature tensor field of the Levi-Civita connection* ∇ *associated to* g. *Moreover,* $\mathcal{D}$ *is involutive,* η *is closed and the Nijenhuis tensor field of* φ *vanishes identically.*

2. Generalized dual connections

Dualistic structures are closely related to statistical mathematics. They consist of pairs of affine connections compatible with a pseudo-Riemannian metric [1]. Their importance in statistical physics is underlined by many authors [8], [19], [21] etc. Also, geometrical applications can be found in affine differential geometry [7], [14], [10], [11].

As a generalization of dual connections, Norden introduced the notion of *generalized dual connection*, which is used in Weyl geometry to characterize Weyl connections [17].

DEFINITION 2.1. *(Norden) Two linear connections $\bar{\nabla}$ and $\bar{\nabla}^*$ are called generalized dual connections with respect to the pseudo-Riemannian metric g by the 1-form η if:*

$$(2.1) \qquad X(g(Y,W)) = g(\bar{\nabla}_X Y, W) + g(Y, \bar{\nabla}^*_X W) + \eta(X)g(Y,W),$$

for any X, Y, $W \in \chi(M)$.

From a direct computation follows that $(\bar{\nabla}^*)^* = \bar{\nabla}$ and

$$(2.2) \qquad g(Y, (\bar{\nabla}^* - \bar{\nabla} + \eta \otimes I)(X,W)) = (\bar{\nabla}_X g)(Y,W),$$

for any X, Y, $W \in \chi(M)$.

In particular, if $\bar{\nabla}$ is a g-metric connection, then its generalized dual $\bar{\nabla}^*$ equals to

$$\bar{\nabla}^* = \bar{\nabla} - \eta \otimes I.$$

PROPOSITION 2.1. *On a para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$, $(\bar{\nabla}, \bar{\nabla}^*)$ satisfy:*

$$g((\bar{\nabla}^*_X \varphi)Y, W) = -g((\bar{\nabla}_X \varphi)W, Y)$$
$$g(\bar{\nabla}^*_X \xi, Y) = g(\bar{\nabla}_X \xi, Y) + (\bar{\nabla}_X g)(Y, \xi) - \eta(X)\eta(Y)$$
$$(\bar{\nabla}^*_X \eta)Y = (\bar{\nabla}_X \eta)Y - (\bar{\nabla}_X g)(Y, \xi) + \eta(X)\eta(Y)$$
$$\bar{\nabla}^* g = -\bar{\nabla} g + 2\eta \otimes g.$$

PROOF. They follow by replacing the expressions

$$(\bar{\nabla}^*_X \varphi)Y := \bar{\nabla}^*_X \varphi Y - \varphi(\bar{\nabla}^*_X Y)$$
$$(\bar{\nabla}^*_X \eta)Y := X(\eta(Y)) - \eta(\bar{\nabla}^*_X Y)$$
$$(\bar{\nabla}^*_X g)(Y, W) := X(g(Y, W)) - g(\bar{\nabla}^*_X Y, W) - g(Y, \bar{\nabla}^*_X W)$$

in (2.1) and taking into account (2.2).   □

In addition, the torsion and the curvature tensor fields of the generalized dual connection $\bar{\nabla}^*$ of the linear connection $\bar{\nabla}$ satisfy:

$$(2.3) \quad T_{\bar{\nabla}^*} = T_{\bar{\nabla}} - (\eta \otimes I - I \otimes \eta), \quad g(R_{\bar{\nabla}^*}(X,Y)W, U) = -g(W, R_{\bar{\nabla}}(X,Y)U).$$

In particular, if $\bar{\nabla}$ is symmetric, then $\bar{\nabla}^*$ is semi-symmetric (as its torsion is of the form $I \otimes \eta - \eta \otimes I$).

Another important feature of the notion of generalized connections is the invariance under gauge transformations. Precisely, $(\bar{\nabla}, \bar{\nabla}^*)$ is a pair of generalized

dual connections with respect to g and η if and only if they are generalized dual connections with respect to $e^f g$ and $\eta - df$, for any smooth function f on M [5].

Remark that different generalizations of the notion of standard dual (or conjugate) connections are considered in [5]: *generalized conjugate connections* [16], [15], *semi-conjugate connections* [16] and *dual semi-conjugate connections* [5], the first arising in Weyl geometry and the second ones in affine hypersurface theory.

3. Canonical connections

In what follows we shall underline the properties of the generalized dual connections of the Levi-Civita connection ∇, of the para-Kenmotsu canonical connection $\tilde{\nabla}$, of the Golab connection ∇^G and of the Zamkovoy canonical paracontact connection ∇^Z associated to the para-Kenmotsu structure (φ, ξ, η, g).

i) The Levi-Civita connection ∇ satisfies [2]:

$$(3.1) \quad \nabla \varphi = g(\varphi \cdot, \cdot) \otimes \xi - \varphi \otimes \eta, \ \ \nabla \xi = I - \eta \otimes \xi, \ \ \nabla \eta = g - \eta \otimes \eta, \ \ \nabla g = 0,$$

its torsion and curvature being given by:

$$(3.2) \quad T_\nabla = 0$$

$$\eta(R_\nabla(X,Y)W) = -\eta(X)g(Y,W) + \eta(Y)g(X,W).$$

Being a g-metric connection, its generalized dual connection ∇^* with respect to g and η is:

$$\nabla^* = \nabla - \eta \otimes I.$$

PROPOSITION 3.1. *On a para-Kenmotsu manifold* $(M, \varphi, \xi, \eta, g)$, *the generalized dual connection* ∇^* *of the Levi-Civita connection* ∇ *satisfies:*

$$\nabla^* \varphi = \nabla \varphi, \ \ \nabla^* \xi = I - 2\eta \otimes \xi, \ \ \nabla^* \eta = \nabla \eta + \eta \otimes \eta, \ \ \nabla^* g = 2\eta \otimes g$$

$$T_{\nabla^*} = -(\eta \otimes I - I \otimes \eta), \ \ R_{\nabla^*}(X,Y)\xi = \eta(X)Y - \eta(Y)X.$$

PROOF. They follow from the relations (3.1), (3.2) and (2.3).   □

ii) The para-Kenmotsu canonical connection $\tilde{\nabla}$ equals to [3]:

$$\tilde{\nabla} := \nabla - I \otimes \eta + g \otimes \xi$$

and satisfies [3]:

$$(3.3) \quad \tilde{\nabla} \varphi = 0, \ \ \tilde{\nabla} \xi = 0, \ \ \tilde{\nabla} \eta = 0, \ \ \tilde{\nabla} g = 0,$$

its torsion and curvature being given by:

$$(3.4) \quad T_{\tilde{\nabla}} = \eta \otimes I - I \otimes \eta$$

$$(3.5) \quad R_{\tilde{\nabla}}(X,Y)W = R_\nabla(X,Y)W - g(W,X)Y + g(Y,W)X - \eta(W)g(X,Y)\xi.$$

Being a g-metric connection, its generalized dual connection $\tilde{\nabla}^*$ with respect to g and η is:

$$\tilde{\nabla}^* = \tilde{\nabla} - \eta \otimes I.$$

PROPOSITION 3.2. *On a para-Kenmotsu manifold* $(M, \varphi, \xi, \eta, g)$, *the general-ized dual connection* $\tilde{\nabla}^*$ *of the para-Kenmotsu canonical connection* $\tilde{\nabla}$ *satisfies:*

$$\tilde{\nabla}^*\varphi = 0, \quad \tilde{\nabla}^*\xi = -\eta \otimes \xi, \quad \tilde{\nabla}^*\eta = \eta \otimes \eta, \quad \tilde{\nabla}^*g = 2\eta \otimes g$$

$$T_{\tilde{\nabla}^*} = 0, \quad R_{\tilde{\nabla}^*}(X, Y)\xi = g(X, Y)\xi.$$

PROOF. They follow from the relations (3.3), (3.4), (3.5) and (2.3). □

iii) The Golab connection ∇^G equals to [**9**]:

$$\nabla^G := \nabla - \eta \otimes \varphi$$

and satisfies [**4**]:

(3.6) $\qquad \nabla^G\varphi = \nabla\varphi, \quad \nabla^G\xi = \nabla\xi, \quad \nabla^G\eta = \nabla\eta, \quad \nabla^G g = \nabla g = 0,$

its torsion and curvature being given by:

(3.7) $\qquad\qquad\qquad T_{\nabla^G} = \varphi \otimes \eta - \eta \otimes \varphi$

(3.8)
$$R_{\nabla^G}(X, Y)W = R_{\nabla}(X, Y)W + g(T, W)\xi - g(\xi, W)T, \quad where \quad T := -T_{\nabla^G}(X, Y).$$

Being a g-metric connection, its generalized dual connection $(\nabla^G)^*$ with respect to g and η is:

$$(\nabla^G)^* = \nabla^G - \eta \otimes I.$$

PROPOSITION 3.3. *On a para-Kenmotsu manifold* $(M, \varphi, \xi, \eta, g)$, *the general-ized dual connection* $(\nabla^G)^*$ *of the Golab connection* ∇^G *satisfies:*

$$(\nabla^G)^*\varphi = \nabla\varphi, \quad (\nabla^G)^*\xi = I - 2\eta \otimes \xi, \quad (\nabla^G)^*\eta = g, \quad (\nabla^G)^*g = 2\eta \otimes g$$

$$T_{(\nabla^G)^*} = -\eta \otimes (\varphi + I) + (\varphi + I) \otimes \eta, \quad R_{(\nabla^G)^*}(X, Y)\xi = \eta(X)(Y - \varphi Y) - \eta(Y)(X - \varphi X).$$

PROOF. They follow from the relations (3.6), (3.7), (3.8) and (2.3). □

iv) The Zamkovoy canonical paracontact connection ∇^Z equals to [**23**]:

$$\nabla^Z_X Y := \nabla_X Y + \eta(X)\varphi Y - \eta(Y)\nabla_X\xi + (\nabla_X\eta)Y \cdot \xi$$

equivalent to:

$$\nabla^Z = \nabla - I \otimes \eta + g \otimes \xi + \eta \otimes \varphi$$

and satisfies [**23**]:

(3.9) $\qquad\qquad \nabla^Z\varphi = 0, \quad \nabla^Z\xi = 0, \quad \nabla^Z\eta = 0, \quad \nabla^Z g = 0,$

its torsion and curvature being given by:

(3.10) $\qquad\qquad T_{\nabla^Z} = \eta \otimes (\varphi + I) - (\varphi + I) \otimes \eta(= -T_{(\nabla^G)^*})$

(3.11) $\qquad\qquad R_{\nabla^Z}(X, Y)W = R_{\nabla}(X, Y)W + g(Y, W)X - g(X, W)Y.$

Being a g-metric connection, its generalized dual connection $(\nabla^Z)^*$ with respect to g and η is:

$$(\nabla^Z)^* = \nabla^Z - \eta \otimes I.$$

PROPOSITION 3.4. *On a para-Kenmotsu manifold* $(M, \varphi, \xi, \eta, g)$, *the generalized dual connection* $(\nabla^Z)^*$ *of the Zamkovoy canonical paracontact connection* ∇^Z *satisfies:*

$$(\nabla^Z)^*\varphi = \nabla\varphi + \varphi \otimes \eta + g(\cdot, \varphi\cdot) \otimes \xi, \quad (\nabla^Z)^*\xi = -\eta \otimes \xi, \quad (\nabla^Z)^*\eta = \eta \otimes \eta,$$

$$(\nabla^Z)^*g = 2\eta \otimes g$$

$$T_{(\nabla^Z)^*} = \eta \otimes \varphi - \varphi \otimes \eta (= -T_{\nabla^G}), \quad R_{(\nabla^Z)^*}(X, Y)\xi = 0.$$

PROOF. They follow from the relations (3.9), (3.10), (3.11) and (2.3).   □

REMARK 3.1. *Remark that the Golab connection* ∇^G *is obtained perturbing the Levi-Civita connection* ∇ *with* $\eta \otimes \varphi$, *so they coincide on* $\mathcal{D}$. *The same thing happens for the para-Kenmotsu canonical connection* $\tilde{\nabla}$ *and the Zamkovoy canonical paracontact connection* ∇^Z. *Therefore:*

$$(\nabla^G)^* = \nabla^* - \eta \otimes \varphi, \quad \tilde{\nabla}^* = (\nabla^Z)^* - \eta \otimes \varphi,$$

the four connections satisfying:

$$\nabla^* + \tilde{\nabla}^* = (\nabla^G)^* + (\nabla^Z)^*.$$

Also notice that if the manifold is of constant curvature, through the four connections, only the Zamkovoy canonical paracontact connection ∇^Z is flat. Indeed, form Proposition 1.1, we deduce that a para-Kenmotsu manifold of constant curvature (i.e. $R_{\nabla}(X, Y)W = k[g(Y, W)X - g(X, W)Y]$, for a constant k) is locally isomorphic to a hyperbolic space (i.e. $k = -1$), therefore $R_{\nabla^Z} = 0$.

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