

ON GENERAL FIRST ZAGREB INDEX OF GRAPHS WITH FIXED MAXIMUM DEGREE¹

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Abstract

The general first Zagreb index M_1^α of a graph G (also called the general zeroth-order Randić index) is equal to the sum of the α -th powers of the vertex degrees of G . For $\alpha > 1$ and $k \geq 0$, we characterize the k -cyclic connected graphs with maximum degree Δ , having minimal general first Zagreb indices.

1 Introduction

In this paper we consider simple graphs, that is graphs without directed, multiple, or weighted edges, and without self-loops. Let G be such a graph, with vertex set $V(G)$ an edge set $E(G)$, possessing $n = |V(G)|$ vertices and $m = |E(G)|$ edges. Throughout this paper it is always assumed that the graphs considered are connected.

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By $d(v) = d_G(v)$ we denote the degree (= number of first neighbors) of the vertex v of the graph G . The maximal vertex degree will be denoted by Δ .

For other graph theoretical terminology and notation we refer to the textbook of Bondy and Murty [3].

In 1972, Trinajstić and one of the present authors [8] introduced a vertex degree-based graph invariant $M_1 = M_1(G)$, that later was named “*first Zagreb index*” [1] and became one of the most thoroughly investigated topological index (for details see the surveys [4, 6, 7, 15] and the references cited therein). It is defined as

$$M_1 = M_1(G) = \sum_{x \in V(G)} d_G(x)^2.$$

In 2004 and 2005, Li et al. [11, 12] introduced the generalized version of first M_1 , defined as

$$M_1^\alpha = M_1^\alpha(G) = \sum_{x \in V(G)} d_G(x)^\alpha$$

where α is a real number. This graph invariant is nowadays known under the name “*general first Zagreb index*” and has also been much investigated (see, for instance [5, 13, 20, 23, 24]).

The very same graph invariant is also being studied in the mathematical literature under the name “*general zeroth-order Randić index*”, see for instance the recent papers [10, 16, 18, 19, 21] and the references cited therein.

For obvious reasons, the parameter α is usually required to be different from 1 and from 0. In what follows, we shall be concerned only with the case $\alpha > 0$. The analysis of the cases $\alpha < 0$ and $0 < \alpha < 1$ is analogous, *mutatis mutandis*.

For a variety of classes of graph, the species with extremal (smallest and greatest) M_1^α -value were determined. However, to the present authors’ knowledge, the case of graphs with a fixed maximum vertex degree has not been investigated. Our aim is to contribute towards filling this gap.

2 Two auxiliary results

Let G be a graph possessing a vertex v of degree one (a pendent vertex) and assume that $G \not\cong P_n$. Then G possesses at least one vertex of degree greater than two. Let u be a vertex of degree greater than two, nearest to

v . Let u' be a first neighbor of u , more distant from v than u . Let the graph G' be obtained by deleting from G the edge uu' and inserting the edge vu' .

Lemma 2.1. *If $\alpha > 1$, then $M_1^\alpha(G) > M_1^\alpha(G')$.*

Proof. G and G' have same vertex degrees, except at v and u , and

$$d_G(v) = 1, \quad d_G(u) = p \geq 3, \quad d_{G'}(v) = 2, \quad d_{G'}(u) = p - 1.$$

Thus,

$$\begin{aligned} M_1^\alpha(G) &= 1^\alpha + p^\alpha + \sum_{x \in V(G) \setminus \{u, v\}} d_G(x)^\alpha \\ M_1^\alpha(G') &= 2^\alpha + (p - 1)^\alpha + \sum_{x \in V(G) \setminus \{u, v\}} d_G(x)^\alpha. \end{aligned}$$

Lemma 2.1 follows now from $1 + p^\alpha > 2^\alpha + (p - 1)^\alpha$, which holds for all $\alpha > 1$ [14]. \square

Let G be a graph possessing two non-adjacent vertices u and v . Let G' be obtained by inserting into G a new edge between u and v .

Lemma 2.2. *If $\alpha > 1$, then $M_1^\alpha(G') > M_1^\alpha(G)$. The difference $M_1^\alpha(G') - M_1^\alpha(G)$ is minimal if both u and v are pendent vertices.*

Proof. G and G' have same vertex degrees, except at u and v , and

$$d_{G'}(u) = d_G(u) + 1, \quad d_{G'}(v) = d_G(v) + 1.$$

Thus,

$$M_1^\alpha(G') - M_1^\alpha(G) = [(d_G(u) + 1)^\alpha - d_G(u)^\alpha] + [(d_G(v) + 1)^\alpha - d_G(v)^\alpha].$$

Lemma 2.2 follows now from the fact that both $d(x)^\alpha$ and $(d(x) + 1)^\alpha - d(x)^\alpha$ are monotonically increasing functions of α . \square

3 General first Zagreb index of trees with maximal degree Δ

A tree is said to be *starlike* if exactly one of its vertices is of degree greater than two [2, 9, 17, 22]. A starlike tree is said to be of degree Δ if its unique

vertex is of degree $\Delta > 2$. Also the path P_n may be considered as a starlike tree of degree $\Delta = 2$.

A starlike tree of order n and of degree Δ has Δ pendent vertices and $n - \Delta - 1$ vertices of degree two. Therefore, all such starlike trees have equal general first Zagreb indices, equal to $\Delta + (n - \Delta - 1)2^\alpha + \Delta^\alpha$.

As a direct consequence of Lemma 2.1, we have:

Theorem 3.1. *If $\alpha > 1$, then among all trees of order n with maximum degree $\Delta \geq 3$, the starlike trees of degree Δ have minimal general first Zagreb index M_1^α .*

Corollary 3.1. *Let $\alpha > 1$. For any tree T of order $n \geq 2$ with maximum degree Δ ,*

$$M_1^\alpha(T) \geq \Delta(\Delta^{\alpha-1} - 2^\alpha + 1) + (n - 1)2^\alpha$$

with equality if and only if T is a starlike.

4 General first Zagreb index of cyclic graphs with maximal degree Δ

Lemma 2.1 implies that cyclic graphs with minimal general first Zagreb index must have as few as possible pendent vertices. Bearing in mind Lemma 2.2 and Theorem 3.1, we also conclude that such minimal graphs should have as many as possible vertices of degree two. This directly yields the following results:

Theorem 4.1. *Let $\alpha > 1$. Then the n -vertex connected unicyclic graphs with maximum degree $\Delta \geq 2$ and with minimal M_1^α -values are those obtained by connecting a pair of pendent vertices of a starlike tree of degree Δ , see Fig. 1.*

Theorem 4.2. *Let $\alpha > 1$. Then the n -vertex connected bicyclic graphs with maximum degree $\Delta \geq 4$ and with minimal M_1^α -values are those obtained by connecting two pairs of pendent vertices of a starlike tree of degree Δ , see Fig. 1.*

Theorem 4.3. *Let $\alpha > 1$. Then the n -vertex connected tricyclic graphs with maximum degree $\Delta \geq 6$ and with minimal M_1^α -values are those obtained by connecting three pairs of pendent vertices of a starlike tree of degree Δ , see Fig. 1.*

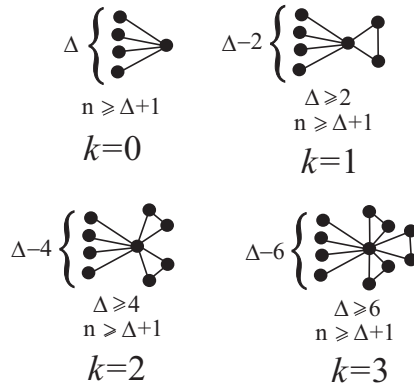


Fig. 1. The k -cyclic graphs with maximum degree $\Delta \geq 2k$ having minimal general first Zagreb indices M_1^α for $k = 0, 1, 2, 3$, and $\alpha > 1$. These possess $n - \Delta + 2k - 1$ vertices of degree 2, which are not indicated. Their position does not influence the value of M_1^α .

The above statements are directly generalized as:

Theorem 4.4. Let $\alpha > 1$ and $k \geq 1$. Then the n -vertex connected k -cyclic graphs with maximum degree $\Delta \geq 2k$ and with minimal M_1^α -values are those obtained by connecting k pairs of pendent vertices of a starlike tree of degree Δ .

Corollary 4.1. Let $\alpha > 1$ and $k \geq 1$. For any connected k -cyclic graph G of order $n \geq 2$ with maximum degree $\Delta \geq 2k$,

$$M_1^\alpha(G) \geq \Delta(\Delta^{\alpha-1} - 2^\alpha + 1) + (n - 1 + 2k)2^\alpha - 2k$$

with equality if and only if G is a graph specified in Theorem 4.4.

If $\Delta < 2k$, then the k -cyclic graphs with minimal M_1^α -values have no pendent vertices and their structure is less straightforward. In what follows we describe the respective exceptional cases for $k = 2$ and $k = 3$.

Proposition 4.1. If $\Delta = 3$, the bicyclic graphs with minimal M_1^α -values are those having two vertices of degree 3 and all other vertices of degree 2. There are two classes of such graphs, depicted in Fig. 2.

If $\Delta = 5$, the tricyclic graphs with minimal M_1^α -values are those having a vertex of degree 5, a vertex of degree 3 and all other vertices of degree 2. There are two classes of such graphs, depicted in Fig. 2.

If $\Delta = 4$, the tricyclic graphs with minimal M_1^α -values are those having a vertex of degree 4, two vertices of degree 3 and all other vertices of degree 2. There are four classes of such graphs, depicted in Fig. 2

If $\Delta = 3$, the tricyclic graphs with minimal M_1^α -values are those having 4 vertices of degree 3 and all other vertices of degree 2. There are four classes of such graphs, depicted in Fig. 2.

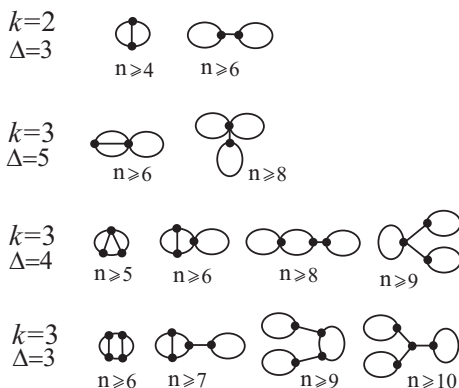


Fig. 2. The classes of k -cyclic graphs with maximum degree $\Delta < 2k$ having minimal general first Zagreb indices M_1^α for $k = 2, 3$, and $\alpha > 1$. The necessary number n of their vertices is indicated.

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