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ON THE LATTICES OF MULTIPLY COMPOSITION FORMATIONS OF FINITE GROUPS

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Abstract. Let τ be a subgroup functor (in Skiba's sense) such that all subgroups of any finite group G contained in $\tau(G)$ are subnormal in G. In this paper, we show first that the lattice of all τ-closed n-multiply composition formations of finite groups $\mathcal{C}_n^\tau$ is not distributive for every non-negative integer n. Next, we prove that if $\mathfrak{F} \neq (1)$ is a non-empty τ-closed n-multiply composition formation then its τ-closed Frattini subformation $\Phi_n^\tau(\mathfrak{F})$ consists of all $\mathcal{C}_n^\tau$-non-generating groups of $\mathfrak{F}$. Moreover, if $\mathfrak{F}_1$ is a τ-closed n-multiple composition subformation of $\mathfrak{F}$ then $\Phi_n^\tau(\mathfrak{F}_1) \subseteq \Phi_n^\tau(\mathfrak{F})$.

1. Introduction

Throughout this paper all groups are finite. So, all group classes considered are subclasses of the class $\mathfrak{G}$ of all finite groups. $\mathfrak{S}$ is the class of all soluble groups. A formation $\mathfrak{F}$ is a class of groups which is closed under homomorphic images and also every group G has smallest normal subgroup with quotient in $\mathfrak{F}$. In the sequel we will consider only subgroup functors τ such that for any group G all subgroups of $\tau(G)$ are subnormal in G. The set of all primes is denoted by $\mathbb{P}$, and p will always denote a prime. For the sake of easy reference, first we cite some concepts and notations from [**1, 3**]. Consider a function

$$f : \mathbb{P} \cup \{0\} \to \{\text{formations of groups}\},$$

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which we call a *composition satellite*. Let G be a group. We denote by $\pi(G)$ the set of all prime divisors of $|G|$. The subgroup $C^p(G)$ is the intersection of the centralizers of all the abelian p-chief factors of G with $C^p(G) = G$ if G has no abelian p-chief factors. For any composition satellite f, we denote by $CLF(f)$ the class of groups G satisfying the following conditions:

(1) $G/R(G) \in f(0)$ where $R(G)$ is the $\mathfrak{S}$-radical $G_{\mathfrak{S}}$ of G;
(2) $G/C^p(G) \in f(p)$ for any prime $p \in \pi(\mathrm{Com}(G))$ where $\mathrm{Com}(G)$ is the class of all simple abelian groups isomorphic to composition factors of G.

The class $CLF(f)$ is a formation, and it is called *composition formation* [**4**].

According to the concept of multiple localisation of composition formations proposed by Skiba and Shemetkov, every formation is 0-multiply composition by definition. For $n > 0$, a formation $\mathfrak{F}$ is called *n-multiply composition* if $\mathfrak{F} = CLF(f)$ and all non-empty values of f are $(n-1)$-multiply composition formations (see [**4**]).

In the universe of all finite groups the definition of a variety leads to the concept of a formation. The lattice of all varieties of groups is modular but is not distributive. The collection of all τ-closed n-multiply composition formations $\mathcal{C}_n^\tau$ is a complete lattice by an inclusion $\subseteq$. Vorob'ev and Tsarev showed that the lattice $\mathcal{C}_n^\tau$ is algebraic and modular (see [**8**, Theorem 3.1]). In Section 3, we prove the following theorem.

Theorem 1.1. *The lattice of all τ-closed n-multiply composition formations $\mathcal{C}_n^\tau$ is not distributive for any non-negative integer n.*

Let $\mathfrak{F}$ and $\mathfrak{H}$ be τ-closed n-multiply composition formations with $\mathfrak{H} \subseteq \mathfrak{F}$. We denote by $\mathfrak{F}/_n^\tau\mathfrak{H}$ the lattice of all τ-closed n-multiply composition formations $\mathfrak{M}$ such as $\mathfrak{H} \subseteq \mathfrak{M} \subseteq \mathfrak{F}$. If $\mathfrak{M} \subset \mathfrak{F}$ and the lattice $\mathfrak{F}/_n^\tau\mathfrak{M}$ consists only two elements then $\mathfrak{M}$ is called *a maximal τ-closed n-multiply composition subformation* of $\mathfrak{F}$ [**4**]. Denote the intersection of all maximal τ-closed n-multiply composition subformations of $\mathfrak{F}$ by $\Phi_n^\tau(\mathfrak{F})$ and call it the *Frattini subformation* of $\mathfrak{F}$. We set $\Phi_n^\tau(\mathfrak{F}) = \mathfrak{F}$ if there are no such subformations.

In the paper [**2**], Guo and Shum extended the Frattini theory of formations and Schunck classes of finite groups to some Frattini theory of formations and Schunck classes of finite universal algebras of Malcev varieties. In Section 4, we establish analogue of [**2**, Theorem 3.1] for τ-closed n-multiple composition formations:

Theorem 1.2. *Let n be non-negative integer and $\mathfrak{F}_1$, $\mathfrak{F}_2$ be non-empty τ-closed n-multiply composition formations. If $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \neq (1)$ then $\Phi_n^\tau(\mathfrak{F}_1) \subseteq \Phi_n^\tau(\mathfrak{F}_2)$.*

All unexplained notations and terminologies are standard. The reader is refereed to [**3, 1**] if necessary.

2. Preliminaries

The concept of subgroup functor turned out to be useful in research on general class theory (see [**3**]). In each group G, we select a system of subgroups $\tau(G)$. We say that τ is a *subgroup functor* if

(1) $G \in \tau(G)$ for every group G;

(2) for every epimorphism $\varphi : A \mapsto B$ and any $H \in \tau(A)$ and $T \in \tau(B)$, we have $H^{\varphi} \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$.

If $\tau(G) = \{G\}$ then the functor τ is called *trivial.* For any collection of groups $\mathfrak{X}$ the symbol S_{τ} denotes the set of groups H such that $H \in \tau(G)$ for some group $G \in \mathfrak{X}$. A class of groups $\mathfrak{F}$ is called τ*-closed* if $\mathrm{S}_{\tau}(\mathfrak{F}) = \mathfrak{F}$. In particular, a formation $\mathfrak{F}$ is called τ*-closed* if $\tau(G) \subseteq \mathfrak{F}$ for every group G of $\mathfrak{F}$.

Let Θ be a set of formations. A formation $\mathfrak{F}$ is called Θ-formation if $\mathfrak{F} \in \Theta$. If the intersection of every set of Θ-formations belongs to Θ and there is a Θ-formation $\mathfrak{H}$ such that $\mathfrak{M} \subseteq \mathfrak{H}$ for every Θ-formation $\mathfrak{M}$ then Θ is called a complete lattice of formations.

Let Θ be a complete lattice of formations. Then $\Theta\mathrm{form}\,\mathfrak{X}$ is the intersection of all Θ-formations containing a class of groups $\mathfrak{X}$. In particular, if $\mathfrak{X} = \{G\}$, we write $\Theta\mathrm{form}\,G$. Any formation of this type is called one-generated Θ-formation. Thus $c_n^{\tau}\mathrm{form}\,\mathfrak{X}$ is the intersection of all τ-closed n-multiply composition formations containing $\mathfrak{X}$, and $c_n^{\tau}\mathrm{form}\,G$ is one-generated τ-closed n-multiply composition formation.

Let $\{f_i \mid i \in I\}$ be a collection of composition satellites. We denote by $\bigcap_{i\in I} f_i$ the composition satellite f such that $f(a) = \bigcap_{i\in I} f_i(a)$ for all $a \in \mathbb{P} \cup \{0\}$.

Lemma 2.1 (Lemma 2 [**4**]). *Let* $\mathfrak{F}_i = CLF(f_i)$, $f = \bigcap_{i\in I} f_i$ *and* $\mathfrak{F} = \bigcap_{i\in I} \mathfrak{F}_i$. *Then* $\mathfrak{F} = CLF(f)$.

Let $\{f_i \mid i \in I\}$ be the collection of all c_{n-1}^{τ}-valued composition satellites of a formation $\mathfrak{F}$. Since the lattice $\mathcal{C}_n^{\tau}$ is complete, using Lemma 2.1, we conclude that $f = \bigcap_{i\in I} f_i$ is a c_{n-1}^{τ}-valued composition satellite of $\mathfrak{F}$. The satellite f is called *minimal.* The following lemma gives a description of the minimal c_{n-1}^{τ}-valued composition satellite of a formation $\mathfrak{F} = c_n^{\tau}\mathrm{form}\mathfrak{X}$.

Lemma 2.2 (Lemma 8 [**6**]). *Let* $n > 0$ *and* $\mathfrak{X}$ *be a non-empty collection of groups. Set* $\pi = \pi(\mathrm{Com}(\mathfrak{X}))$, *and let* f *be the minimal* c_{n-1}^{τ}*-valued composition satellite of the formation* $\mathfrak{F} = c_n^{\tau}\mathrm{form}\,\mathfrak{X}$. *Then:*

(1) $f(0) = c_{n-1}^{\tau}\mathrm{form}(G/R(G) \mid G \in \mathfrak{X})$;
(2) $f(p) = c_{n-1}^{\tau}\mathrm{form}(G/C^p(G) \mid G \in \mathfrak{X})$ *for all* $p \in \pi$;
(3) $f(p) = \varnothing$ *for all* $p \notin \pi$;
(4) *if* $\mathfrak{F} = CLF(h)$ *where* h *is a* c_{n-1}^{τ}*-valued composition satellite then*

$$f(0) = c_{n-1}^{\tau}\mathrm{form}(G \mid G \in h(0) \cap \mathfrak{F},\, R(G) = 1),$$

and for all $p \in \pi$ *we have*

$$f(p) = c_{n-1}^{\tau}\mathrm{form}(G \mid G \in h(p) \cap \mathfrak{F},\, O_p(G) = 1).$$

By Lemma 2.2 it is easy to show the following assertion

Lemma 2.3. *Let* $n > 0$, f_1 *and* f_2 *be the minimal composition* c_{n-1}^{τ}*-valued satellites of formations* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *respectively. Then* $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ *if and only if* $f_1 \leqslant f_2$.

Let Θ be a complete lattice of formations. We denote

$$\vee_{\Theta}(\mathfrak{F}_i \mid i \in I) = \Theta\mathrm{form}\left(\bigcup_{i\in I}\mathfrak{F}_i\right) \text{ and } \vee_n^{\tau}(\mathfrak{F}_i \mid i \in I) = c_n^{\tau}\mathrm{form}\left(\bigcup_{i\in I}\mathfrak{F}_i\right) \text{ if } \Theta = \mathcal{C}_n^{\tau}$$

for a collection of Θ-formations $\{\mathfrak{F}_i \mid i \in I\}$. Let $\{f_i \mid i \in I\}$ be a collection of Θ-valued composition satellites f_i. In this case, by $\vee_{\Theta}(f_i \mid i \in I)$ we denote a function f such that $f(a) = \Theta\mathrm{form}(\cup_{i\in I} f_i(a))$ for all $a \in \mathbb{P} \cup \{0\}$. We denote by Θ^c the set of all formations having a Θ-valued composition satellite. A complete lattice Θ^c is called *inductive* if for any collection $\{\mathfrak{F}_i = CLF(f_i) \mid i \in I\}$, where f_i is an integrated satellite of $\mathfrak{F}_i \in \Theta^c$, the equality $\vee_{\Theta^c}(\mathfrak{F}_i \mid i \in I) = CLF(\vee_{\Theta}(f_i \mid i \in I))$ holds.

LEMMA 2.4 (Theorem [**6**]). *The lattice $\mathcal{C}_n^{\tau}$ is inductive.*

The inductance of a lattice $\mathcal{C}_n^{\tau}$ means that a research of the operation $\vee_n^{\tau}$ can be reduced to research of the operation $\vee_{n-1}^{\tau}$. For every term ξ of signature $\{\cap, \vee_n^{\tau}\}$ we denote by $\overline{\xi}$ the term of signature $\{\cap, \vee_{n-1}^{\tau}\}$ obtained from ξ by replacing of every symbol $\vee_n^{\tau}$ to the symbol $\vee_{n-1}^{\tau}$.

LEMMA 2.5. *Let Θ^c be an inductive lattice of formations, $\xi(x_{i_1}, \dots, x_{i_m})$ be a term of signature $\{\cap, \vee_{\Theta^c}\}$ and f_i be an inner Θ-valued composition satellite of a formation $\mathfrak{F}_i$ where $i = 1, \dots, m$. Then $\xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m) = CLF(\overline{\xi}(f_1, \dots, f_m))$.*

PROOF. See the proof of [**3**, Lemma 4.2.1]. □

A complete lattice of formations Θ is called $\mathfrak{G}$-*separated* [**3**, p. 159] if for any term $\xi(x_1, \dots, x_m)$ of signature $\{\cap, \vee_{\Theta}\}$, any formations $\mathfrak{F}_1, \dots, \mathfrak{F}_m$ of Θ, and any group $A \in \xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m)$, there exist groups $A_1 \in \mathfrak{F}_1, \dots, A_m \in \mathfrak{F}_m$ such that $A \in \xi(\Theta\mathrm{form}\, A_1, \dots, \Theta\mathrm{form}\, A_m)$.

LEMMA 2.6 (Lemma 17 [**7**]). *Let Θ be a $\mathfrak{G}$-separated lattice of formations and let η be a sublattice of Θ such that η contains all one-generated Θ-subformations of the form $\Theta\mathrm{form}\, A$ of every formation $\mathfrak{F} \in \eta$. It is supposed that a law $\xi_1 = \xi_2$ of signature $\{\cap, \vee_{\Theta}\}$ is true for all one-generated Θ-formations belonging to η. Then the law $\xi_1 = \xi_2$ is true for all Θ-subformations belonging to η.*

3. Main Results

3.1. Proof of Theorem 1.1.

LEMMA 3.1. *Let $n > 0$ and $\mathfrak{F}_i = c_n^{\tau}\mathrm{form}(Z_p \wr A_i)$ where $p \notin \pi(A_i)$ for $i = 1, 2$. Then $f(p) = f_1(p) \cap f_2(p)$ where f_i and f are the minimal c_{n-1}^{τ}-valued composition satellites of the formations $\mathfrak{F}_i$ and $\mathfrak{F} = \mathfrak{F}_1 \cap \mathfrak{F}_2$, respectively.*

PROOF. Let $h = f_1 \cap f_2$. By Lemma 2.1 $\mathfrak{F} = CLF(h)$. Let $p \notin \pi(A_1, A_2)$ then $O_p(A_1) = 1 = O_p(A_2)$. By Lemma 2.2 $f(p) = c_{n-1}^{\tau}\mathrm{form}(A \mid A \in h(p), O_p(A) = 1)$. Set $B_i = Z_p \wr A_i$. Then by [**5**, Lemma 2.8] $B_i/C^p(B_i) = B_i/O_p(B_i) \cong A_i$ where $i = 1, 2$. Hence by Lemma 2.2 $f_i(p) = c_{n-1}^{\tau}\mathrm{form}(B_i/C^p(B_i)) = c_{n-1}^{\tau}\mathrm{form}A_i$. Since $A_i \in \mathfrak{G}_{p'}$ we have $f_i(p) = c_{n-1}^{\tau}\mathrm{form}A_i \subseteq \mathfrak{G}_{p'}$. Hence $O_p(A) = 1$ for any group $A \in f_i(p)$. Consequently $f(p) = c_{n-1}^{\tau}\mathrm{form}(f_1(p) \cap f_2(p)) = h(p)$. □

Lemma 3.2. *Let* $n > 0$ *and* $\mathfrak{F}_i = c_n^\tau\mathrm{form}(Z_p \wr A_i)$ *where* $p \notin \pi(A_i)$ *for* $i = 1, \dots, m$. *Let* f_i *be the minimal* c_{n-1}^τ*-valued composition satellite of* $\mathfrak{F}_i$ *and*

$$f(p) = \overline{\xi}(f_1, \dots, f_m)(p)$$

where $\xi(x_1, \dots, x_m)$ *is a term of signature* $\{\cap, \vee_n^\tau\}$. *Then* f *is the minimal* c_{n-1}^τ*-valued composition satellite of* $\mathfrak{F} = \xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m)$.

Proof. Set $h = \overline{\xi}(f_1, \dots, f_m)$. By Lemmas 2.4 and 2.5 $\xi(\mathfrak{F}_1, \dots, \mathfrak{F}_m) = CLF(h)$. We show that $h(p) = f(p)$ by induction on the number r of occurrences of the symbols in $\{\cap, \vee_n^\tau\}$ into ξ. The case $r = 1$ we have using [**6**, Lemma 11] and Lemma 3.1.

Let the term ξ have $r > 1$ occurrences of the symbols in $\{\cap, \vee_n^\tau\}$. Let ξ have the form

$$\xi(x_1, \dots, x_m) = \xi_1(x_{i_1}, \dots, x_{i_a}) \triangle \xi_2(x_{j_1}, \dots, x_{j_b})$$

where $\{x_{i_1}, \dots, x_{i_a}\} \cup \{x_{j_1}, \dots, x_{j_b}\} = \{x_1, \dots, x_m\}$, $\triangle \in \{\cap, \vee_n^\tau\}$, and the assertion is true for ξ_1 and ξ_2. By induction $h_1(p) = \overline{\xi}_1(f_{i_1}, \dots, f_{i_a})(p)$ and $h_2(p) = \overline{\xi}_2(f_{j_1}, \dots, f_{j_b})(p)$ where h_1 and h_2 are the minimal c_{n-1}^τ-valued composition satellites of $\xi_1(\mathfrak{F}_{i_1}, \dots, \mathfrak{F}_{i_a})$ and $\xi_2(\mathfrak{F}_{j_1}, \dots, \mathfrak{F}_{j_b})$, respectively. Thus

$$f(p) = h_1(p) \,\overline{\triangle}\, h_2(p) = \overline{\xi}_1(f_{i_1}(p), \dots, f_{i_a}(p)) \,\overline{\triangle}\, \overline{\xi}_2(f_{j_1}(p), \dots, f_{j_b}(p)) =$$

$$= \overline{\xi}(f_1(p), \dots, f_m(p)) = \overline{\xi}(f_1, \dots, f_m)(p) = h(p),$$

as claimed. □

Proof of Theorem 1.1. We proceed by induction on n. By [**7**, Corollary 5] the lattice of all τ-closed formations $\mathcal{C}_0^\tau$ is not distributive.

Let $n > 0$ and let the lattice $\mathcal{C}_{n-1}^\tau$ be not distributive. The lattice $\mathcal{C}_n^\tau$ is supposed to be distributive. Let A_1, A_2, A_3 be groups. Let $\mathfrak{F}_i = c_n^\tau\mathrm{form}B_i$ $(i = 1, 2, 3)$ where $B_i = Z_p \wr A_i$, $p \notin \pi = \pi(A_1, A_2, A_3)$. Then

$$\mathfrak{F} = \mathfrak{F}_1 \cap (\mathfrak{F}_2 \vee_n^\tau \mathfrak{F}_3) = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \vee_n^\tau (\mathfrak{F}_1 \cap \mathfrak{F}_3) = \mathfrak{H}.$$

Let f, h and f_i be the minimal c_{n-1}^τ-valued composition satellites of $\mathfrak{F}$, $\mathfrak{H}$ and $\mathfrak{F}_i$, respectively. By Lemma 2.3 $f(p) = h(p)$. Since $p \notin \pi$ then $O_p(A_i) = 1$ for all $i = 1, 2, 3$. By [**5**, Lemma 2.8] we have

$$B_i/C^p(B_i) = B_i/O_p(B_i) \cong A_i.$$

By Lemma 2.2 $f_i(p) = c_{n-1}^\tau\mathrm{form}(B_i/C^p(B_i)) = c_{n-1}^\tau\mathrm{form}A_i$. Using Lemma 3.2 we obtain

$$f(p) = c_{n-1}^\tau\mathrm{form}\, A_1 \cap \Big((c_{n-1}^\tau\mathrm{form}\, A_2) \vee_{n-1}^\tau (c_{n-1}^\tau\mathrm{form}\, A_3) \Big) =$$

$$= \Big(c_{n-1}^\tau\mathrm{form}\, A_1 \cap c_{n-1}^\tau\mathrm{form}\, A_2 \Big) \vee_{n-1}^\tau \Big(c_{n-1}^\tau\mathrm{form}\, A_1 \cap c_{n-1}^\tau\mathrm{form}\, A_3 \Big) = h(p).$$

By [**7**, Proposition] the lattice of all n-multiply composition formations is $\mathfrak{G}$-separated. Consequently, applying Lemma 2.6, we conclude that the lattice $\mathcal{C}_{n-1}^\tau$ is distributive for trivial functor τ, a contradiction. This proves the theorem. □

3.2. Proof of Theorem 1.2.

REMARK 3.1. Let G be a group and $\mathfrak{X}$ be a non-empty collection of groups. Then the formation $\mathfrak{F} = c_n^\tau\text{form}(\mathfrak{X} \cup \{G\})$ contains a maximal $\mathcal{C}_n^\tau$-subformation containing $c_n^\tau\text{form}\,\mathfrak{X} \neq \mathfrak{F}$ for any non-negative integer n.

PROOF. We denote by Ω be the partially ordered collection of all $\mathcal{C}_n^\tau$-subformations of $\mathfrak{F}$ that contain $c_n^\tau\text{form}\,\mathfrak{X}$ but do not contain G. Let $\{\mathfrak{F}_i \mid i \in I\}$ be a chain from Ω. Set $\mathfrak{H} = \bigcup_{i\in I} \mathfrak{F}_i$. Then by [**4**, Lemma 3] $\mathfrak{H}$ is an n-multiply composition formation, and by [**8**, Lemma 3.1] it is τ-closed. We note that $c_n^\tau\text{form}\,\mathfrak{X} \subseteq \mathfrak{H}$ and $G \notin \mathfrak{H}$. By the Kuratowski–Zorn lemma every element $\mathfrak{M} \in \Omega$ is contained in a maximal element from Ω.

Now we show that $\mathfrak{M}$ is a maximal $\mathcal{C}_n^\tau$-subformation of $\mathfrak{F}$. Let $\mathfrak{L}$ be a $\mathcal{C}_n^\tau$-formation such that $\mathfrak{M} \subset \mathfrak{L} \subset \mathfrak{F}$. Since $\mathfrak{X} \subseteq c_n^\tau\text{form}\,\mathfrak{X} \subseteq \mathfrak{M} \subset \mathfrak{L}$, we obtain $G \notin \mathfrak{L}$. Therefore, $\mathfrak{L}$ belongs to Ω. This contradicts the choice of $\mathfrak{M}$. The result follows. □

Let $\mathfrak{X}$ be a non-empty collection of groups. If $\mathfrak{F} = c_n^\tau\text{form}\,(\mathfrak{X} \cup \{G\})$ always implies that $\mathfrak{F} = c_n^\tau\text{form}\,\mathfrak{X}$ then G is called $\mathcal{C}_n^\tau$*-non-generating group* of the formation $\mathfrak{F}$ [**4**].

REMARK 3.2. Let $n > 0$ and $\mathfrak{F} \neq (1)$ be non-empty τ-closed n-multiply composition formation. Then $\Phi_n^\tau(\mathfrak{F})$ consists of all $\mathcal{C}_n^\tau$-non-generating groups of $\mathfrak{F}$.

PROOF. Let G and $\mathfrak{L}$ be $\mathcal{C}_n^\tau$-non-generating group and a maximal $\mathcal{C}_n^\tau$-subformation of $\mathfrak{F}$ respectively. It is assumed that $G \notin \mathfrak{L}$. Then

$$c_n^\tau\text{form}(\mathfrak{L} \cup \{G\}) = \mathfrak{F} = c_n^\tau\text{form}\,\mathfrak{L} = \mathfrak{L},$$

which is a contradiction. Thus $G \in \mathfrak{L}$.

Let $\mathfrak{X}$ be a non-empty collection of groups contained in $\mathfrak{F}$ and $G \in \Phi_n^\tau(\mathfrak{F})$. It is assumed that $c_n^\tau\text{form}(\mathfrak{X} \cup \{G\}) = \mathfrak{F} \neq c_n^\tau\text{form}\,\mathfrak{X}$. By Remark 3.1 $\mathfrak{F}$ contains a maximal $\mathcal{C}_n^\tau$-subformation $\mathfrak{M}$ such that $c_n^\tau\text{form}\,\mathfrak{X} \subseteq \mathfrak{M}$. Thus, since $G \in \Phi_n^\tau(\mathfrak{F})$, forcing $\mathfrak{M}$ to be equal to $\mathfrak{F}$, contradicting the choices of this formations. Consequently, $\mathfrak{F} = c_n^\tau\text{form}\,\mathfrak{X}$. □

The lattice $\mathcal{C}_n^\tau$ is modular by [**8**]. As an immediate corollary from this fact we obtain the following result.

LEMMA 3.3. *For any two τ-closed n-multiply composition formations $\mathfrak{M}$ and $\mathfrak{F}$ the lattices $(\mathfrak{M} \vee_n^\tau \mathfrak{F})/_n^\tau\mathfrak{M}$ and $\mathfrak{F}/_n^\tau(\mathfrak{F} \cap \mathfrak{M})$ are isomorphic.*

Now we are ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Set $\Phi_i = \Phi_n^\tau(\mathfrak{F}_i)$ where $i = 1, 2$. Suppose that $\Phi_1 \not\subseteq \Phi_2$. Then a maximal $\mathcal{C}_n^\tau$-subformation $\mathfrak{M}$ of $\mathfrak{F}_2$ with $\Phi_1 \not\subseteq \mathfrak{M}$ exists there. Thus $\mathfrak{F}_1 \not\subseteq \mathfrak{M}$. By Lemma 3.3 we have

$$\mathfrak{F}_2/_n^\tau\mathfrak{M} = (\mathfrak{M} \vee_n^\tau \mathfrak{F}_1)/_n^\tau\mathfrak{M} \simeq \mathfrak{F}_1/_n^\tau(\mathfrak{F}_1 \cap \mathfrak{M}).$$

The lattice from the left side consists only two elements. Then $\mathfrak{F}_1 \cap \mathfrak{M}$ is the maximal $\mathcal{C}_n^\tau$-subformation of $\mathfrak{F}_1$. Hence $\Phi_1 \subseteq \mathfrak{M}$. We obtain a contradiction. Consequently $\Phi_1 \subseteq \Phi_2$, as asserted. □

Corollary 3.1. *Let n be non-negative integer and $\mathfrak{F}$ be soluble τ-closed n-multiply composition formation with $\Phi_n^\tau(\mathfrak{F}) = (1)$. Then $\mathfrak{F} \subseteq \mathfrak{N}$.*

4. Concluding remarks

We note that one-generated τ-closed n-multiply composition formations are compact elements of the algebraic lattice $\mathcal{C}_n^\tau$ (see proof of [**8**, Theorem 3.1]). For every one-generated τ-closed n-multiply composition formation $\mathfrak{F}$, we have $\Phi_n^\tau(\mathfrak{F}) \neq \mathfrak{F}$. Therefore, the case where $\Phi_n^\tau(\mathfrak{F})$ coincides with a maximal $\mathcal{C}_n^\tau$-subformation of $\mathfrak{F}$ is of special interest. Let $\mathfrak{F}$ be non-empty τ-closed n-multiply composition formation and let $\{\mathfrak{F}_i \mid i \in I\}$ be the collection of all $\mathcal{C}_n^\tau$-subformation of $\mathfrak{F}$. If $\vee_n^\tau(\mathfrak{F}_i \mid i \in I) \subset \mathfrak{F}$ then $\mathfrak{F}$ called an *irreducible* $\mathcal{C}_n^\tau$-formation. If $\vee_n^\tau(\mathfrak{F}_i \mid i \in I) = \mathfrak{F}$ then $\mathfrak{F}$ called a *reducible* $\mathcal{C}_n^\tau$-formation.

Remark 4.1. Let $\mathfrak{F}$ be an irreducible $\mathcal{C}_n^\tau$-formation for non-negative integer n. Then $\mathfrak{F}$ is one-generated $\mathcal{C}_n^\tau$-formation with only one maximal $\mathcal{C}_n^\tau$-subformation.

Let $\mathfrak{F}$ be non-empty τ-closed n-multiply composition formation and $\mathfrak{H}$ be a class of groups. If $\mathfrak{F} \not\subseteq \mathfrak{H}$ and $\mathfrak{L} \subseteq \mathfrak{H}$ for every $\mathcal{C}_n^\tau$-subformation $\mathfrak{L}$ of $\mathfrak{F}$ then $\mathfrak{F}$ called a *minimal* τ-closed n-multiply composition $\mathfrak{H}$-formation.

Remark 4.2. Let n be non-negative integer, $\mathfrak{H} \in \mathcal{C}_n^\tau$ and $\mathfrak{F}$ be a minimal τ-closed n-multiply composition $\mathfrak{H}$-formation. Then $\mathfrak{F}$ is an irreducible $\mathcal{C}_n^\tau$-formation with $\Phi_n^\tau(\mathfrak{F}) \subseteq \mathfrak{H}$.

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