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# Coincidence and Fixed point of Nonexpansive type Mappings in 2-Metric Spaces

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Abstract. The aim of this paper is to prove a coincidence point theorem for a class of self mappings satisfying nonexpansive type condition under various conditions and a fixed point theorem is also obtained. Our results extend and generalizes the corresponding result of Singh et al. [**7**].

## 1. INTRODUCTION AND PRELIMINARIES

The concept of 2-metric space was introduced by Gähler ([**2, 3, 4**]) whose abstract properties were suggested by the area function in Euclidean space. Employing various contractive conditions Iseki [**5**] setout the tradition of proving fixed point theorems in 2-metric spaces. Later on, Naidu and Prasad [**6**] contributed few fixed point theorems in 2-metric spaces introducing the concept of weak commutativity.

Recently, Singh et al. sg proved a fixed point theorem in 2-metric space for nonexpansive type mappings. They obtained the following result:

Theorem 1.1. *Let $(X, d)$ be a 2-metric space and $T : X \to X$ be a self mapping satisfying the following nonexpansive type condition:*

$$\begin{aligned} d(Tx, Ty, u) \leq \\ & a \max\{d(x, y, u), d(x, Tx, u), d(y, Ty, u), \frac{1}{2}[d(x, Ty, u) + d(y, Tx, u)]\} \\ & + b \max\{d(x, Tx, u), d(y, Ty, u)\} + c[d(x, Ty, u) + d(y, Tx, u)] \end{aligned} \tag{1.1}$$

*for all $x, y, u \in X$, where $a, b, c$ are real numbers such that $a + b + 2c = 1$ and $a \geq 0,\ b > 0,\ c > 0$. Then $T$ has a unique fixed point and $T$ is continuous at the fixed point.*

In this paper, we introduce a new class of self mappings satisfying the following nonexpansive type condition:

$$\begin{aligned} d(Tx, Ty, u) \leq\ & a(x, y) \max\{d(fx, fy, u), d(fy, Ty, u)\} \\ & + b \max\{d(fx, Tx, u), d(fy, Ty, u), d(y, Tx, u)\} \\ & + c[d(fx, Ty, u) + d(fy, Tx, u)] \end{aligned} \tag{1.2}$$

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for all $x, y, u \in X$, where $a, b, c$ are real numbers such that $\sup\{a(x,y) + b(x,y) + 2c(x,y)\} = 1$ and $a(x,y) \geq 0$, $\beta = \inf b(x,y) > 0$, $\gamma = \inf c(x,y) > 0$. Our condition is an extension of that of Ćirić ([**1**])(see also [**8**]).

Also, we will show that our condition (1.2) includes the above condition (1.1) of S. L. Singh et al. [**7**].

Now we give some definitions which are used frequently to prove our main results.

Gähler defined 2-metric space as follows:

Definition 1.1. *A 2-metric on a set $X$ with at least three points is a non-negative real-valued mapping $d : X \times X \times X \to R$ satisfying the following properties:*

(1) *To each pair of points $a$, $b$ with $a \neq b$ in $X$ there is a point $c \in X$ such that $d(a,b,c) \neq 0$.*
(2) $d(a,b,c) = 0$, *if at least two of the points are equal,*
(3) $d(a,b,c) = d(b,c,a) = d(a,c,b)$,
(4) $d(a,b,c) \leq (a,b,u) + d(a,u,c) + d(u,b,c)$ *for all $a, b, c, u \in X$.*

*The pair $(X, d)$ is called a 2-metric space.*

Definition 1.2. *The sequence $\{x_n\}$ is convergent to $x \in X$ and $x$ is the limit of this sequence if $\lim_{n\to\infty} d(x_n, x, u) = 0$ for each $u \in X$.*

Definition 1.3. *A sequence $\{x_n\}$ is called Cauchy sequence if*

$$\lim_{n,m\to\infty} d(x_n, x_m, u) = 0$$

*for all $u \in X$. A 2-metric space in which every Cauchy sequence is convergent is called complete.*

Definition 1.4. *Let $f$ and $g$ be two self mappings of a 2-metric space $(X, d)$. Then $f$ and $g$ are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n, u) = 0$ for each $u \in X$, whenever $\{x_n\}$ is a sequence such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X.$$

## 2. MAIN RESULTS

Theorem 2.1. *Let $(X, d)$ be a 2-metric space. Let $T$, $f$ be self mappings of $X$ satisfying nonexpansive type condition (1.2) with $\sup\{a(x,y)+b(x,y)+2c(x,y)\} = 1$ and $a(x,y) \geq 0$, $\beta = \inf b(x,y) > 0$, $\gamma = \inf c(x,y) > 0$. Let $T(X) \subseteq f(X)$ and either*
**(a)** *$X$ is complete and $f$ is surjective, or,*
**(b)** *$X$ is complete, $f$ is continuous and $T$, $f$ are compatible, or*
**(c)** *$f(X)$ is complete, or*
**(d)** *$T(X)$ is complete.*
*Then $f$ and $T$ have a coincidence point in $X$ . Further, the coincidence point is unique, that is, $f_p = f_q$, whenever $f_p = T_p$ and $f_q = T_q$; $p, q \in X$.*

Proof. Let $x = x_0$ be an arbitrary point in $X$. Since $T(X) \subseteq f(X)$, choose $x_1$ so that $y_1 = fx_1 = Tx_0$. In general, choose $x_{n+1}$ such that $y_{n+1} = fx_{n+1} = Tx_n$ for all $n = 0, 1, 2, \cdots$.
On applying inequality (1.2) and taking $a(x_n, x_{n+1}) = a$, $b(x_n, x_{n+1}) = b$ and

$c(x_n, x_{n+1}) = c$, we get

$$\begin{aligned} d(fx_{n+2}, fx_{n+1}, fx_n) &= d(Tx_{n+1}, Tx_n, fx_n) \\ &\leq a \max\{d(fx_{n+1}, fx_n, fx_n), d(fx_{n+1}, Tx_{n+1}, fx_n)\} \\ &\quad + b \max\{d(fx_n, Tx_n, fx_n), d(fx_{n+1}, Tx_{n+1}, fx_{n+1}) \\ &\qquad , d(fx_{n+1}, Tx_n, fx_n)\} \\ &\quad + c[d(fx_n, Tx_{n+1}, fx_n) + d(fx_{n+1}, Tx_n, fx_n)] \\ &= (a+b)d(fx_{n+1}, Tx_{n+1}, fx_n) \\ &= (a+b)d(fx_{n+2}, fx_{n+1}, fx_n) \end{aligned}$$

This implies that

$$(1-(a+b))d(fx_{n+2}, fx_{n+1}, fx_n) \leq 0$$

Since $1-(a+b) > 0$, we get

$$d(fx_{n+2}, fx_{n+1}, fx_n) = 0 \tag{2.1}$$

On applying inequality (1.2) again and using triangular inequality and (2.1), we get

$$\begin{aligned} d(Tx_n, Tx_{n+1}, u) &\leq a \max\{d(fx_n, fx_{n+1}, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &\quad + b \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u) \\ &\qquad , d(fx_{n+1}, Tx_n, u)\} \\ &\quad + c[d(fx_n, Tx_{n+1}, u) + d(fx_{n+1}, Tx_n, u)] \\ &\leq a \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &\quad + b \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &\quad + cd(fx_n, Tx_{n+1}, u) \\ &= a \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &\quad + b \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &\quad + c[d(fx_n, Tx_{n+1}, Tx_n) + d(fx_n, Tx_n, u) \\ &\qquad + d(Tx_{n+1}, Tx_n, u)] \\ &= a \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &\quad + b \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &\quad + c[d(fx_n, Tx_n, u) + d(fx_{n+1}, Tx_{n+1}, u)] \end{aligned} \tag{2.2}$$

Suppose that, for some $n$, $d(fx_{n+1}, Tx_{n+1}, u) > d(fx_n, Tx_n, u)$, then from (2.2),we have

$$\begin{aligned} d(fx_{n+1}, Tx_{n+1}, u) &= d(Tx_n, Tx_{n+1}, u) \\ &\leq ad(fx_{n+1}, Tx_{n+1}, u) + bd(fx_{n+1}, Tx_{n+1}, u) \\ &\quad + c[d(fx_{n+1}, Tx_{n+1}, u) + d(fx_{n+1}, Tx_{n+1}, u)] \\ &= (a+b+2c)d(fx_{n+1}, Tx_{n+1}, u) \\ &\leq d(fx_{n+1}, Tx_{n+1}, u) \end{aligned}$$

a contradiction. Hence we must have, $d(fx_{n+1}, Tx_{n+1}, u) \leq d(fx_n, Tx_n, u)$,or equivalently,

$$d(Tx_n, Tx_{n+1}, u) \leq d(Tx_{n-1}, Tx_n, u) \tag{2.3}$$

On applying inequality (1.2) again and evaluating $a,\ b,\ c$ at $(x_{n-1},x_n)$, we have

$$\begin{aligned} d(y_n,y_{n+1},u) &= d(Tx_{n-1},Tx_n,u)\\ &\le a\max\{d(fx_{n-1},fx_n,u),d(fx_n,Tx_n,u)\}\\ &\quad + b\max\{d(fx_{n-1},Tx_{n-1},u),d(fx_n,Tx_n,u)\\ &\qquad ,d(fx_n,Tx_{n-1},u)\}\\ &\quad + c[d(fx_{n-1},Tx_n,u)+d(fx_n,Tx_{n-1},u)]\\ &= a\max\{d(Tx_{n-2},Tx_{n-1},u),d(Tx_{n-1},Tx_n,u)\}\\ &\quad b\max\{d(Tx_{n-2},Tx_{n-1},u),d(Tx_{n-1},Tx_n,u)\}\\ &\quad + cd(Tx_{n-2},Tx_n,u)\\ &= ad(Tx_{n-2},Tx_{n-1},u)+bd(Tx_{n-2},Tx_{n-1},u)\\ &\quad + cd(Tx_{n-2},Tx_n,u) \end{aligned} \tag{2.4}$$

On applying inequality (1.2) again and using (2.1), (2.3) and by triangular inequality, we get

$$\begin{aligned} d(Tx_{n-2},Tx_n,u) &\le \overline{a}\max\{d(fx_{n-2},fx_n,u),d(fx_n,Tx_n,u)\}\\ &\quad + \overline{b}\max\{d(fx_{n-2},Tx_{n-2},u),d(fx_n,Tx_n,u)\\ &\qquad ,d(fx_n,Tx_{n-2},u)\}\\ &\quad + \overline{c}[d(fx_{n-2},Tx_n,u)+d(fx_n,Tx_{n-2},u)]\\ &= \overline{a}\max\{d(Tx_{n-3},Tx_{n-1},u),d(Tx_{n-1},Tx_n,u)\}\\ &\quad + \overline{b}\max\{d(Tx_{n-3},Tx_{n-2},u),d(Tx_{n-1},Tx_n,u)\\ &\qquad ,d(Tx_{n-1},Tx_{n-2},u)\}\\ &\quad + \overline{c}[d(Tx_{n-3},Tx_n,u)+d(Tx_{n-1},Tx_{n-2},u)] \end{aligned}$$

$$\begin{aligned} &\le \overline{a}\max\{d(Tx_{n-3},Tx_{n-2},Tx_{n-1})+d(Tx_{n-3},Tx_{n-2},u)\\ &\qquad + d(Tx_{n-2},Tx_{n-1},u),d(Tx_{n-1},Tx_n,u)\}\\ &\quad + \overline{b}\max\{d(Tx_{n-3},Tx_{n-2},u),d(Tx_{n-1},Tx_n,u)\\ &\quad ,d(Tx_{n-1},Tx_{n-2},u)\}\\ &\quad + \overline{c}[d(Tx_{n-3},Tx_{n-2},Tx_n)+d(Tx_{n-3},Tx_{n-2},u)\\ &\qquad + d(Tx_{n-2},Tx_n,u)+d(Tx_{n-1},Tx_{n-2},u)] \end{aligned}$$

$$\begin{aligned} &\le \overline{a}\max\{d(Tx_{n-3},Tx_{n-2},Tx_{n-1})+d(Tx_{n-3},Tx_{n-2},u)\\ &\qquad + d(Tx_{n-2},Tx_{n-1},u),d(Tx_{n-1},Tx_n,u)\}\\ &\quad + \overline{b}\max\{d(Tx_{n-3},Tx_{n-2},u),d(Tx_{n-1},Tx_n,u)\\ &\quad ,d(Tx_{n-1},Tx_{n-2},u)\}\\ &\quad + \overline{c}[d(Tx_{n-3},Tx_{n-2},Tx_{n-1})+d(Tx_{n-3},Tx_{n-1},Tx_n)\\ &\qquad + d(Tx_{n-2},Tx_{n-1},Tx_n)+d(Tx_{n-3},Tx_{n-2},u)\\ &\qquad + d(Tx_{n-2},Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n-1},u)\\ &\qquad + d(Tx_{n-2},Tx_{n-1},u)+d(Tx_{n-1},Tx_{n-2},u)] \end{aligned}$$

$$
\begin{aligned}
&= \overline{a}\max\{d(Tx_{n-3},Tx_{n-2},u)+d(Tx_{n-2},Tx_{n-1},u) \\
&\quad ,d(Tx_{n-1},Tx_n,u)\} \\
&\quad + \overline{b}\max\{d(Tx_{n-3},Tx_{n-2},u),d(Tx_{n-1},Tx_n,u) \\
&\quad ,d(Tx_{n-1},Tx_{n-2},u)\} \\
&\quad + \overline{c}[d(Tx_{n-3},Tx_{n-1},Tx_n)+d(Tx_{n-3},Tx_{n-2},u) \\
&\quad + d(Tx_n,Tx_{n-1},u)+d(Tx_{n-2},Tx_{n-1},u) \\
&\quad + d(Tx_{n-1},Tx_{n-2},u)] \\
&\le \overline{a}\max\{2d(Tx_{n-3},Tx_{n-2},u),d(Tx_{n-3},Tx_{n-2},u)\} \\
&\quad + \overline{b}\max\{d(Tx_{n-3},Tx_{n-2},u),d(Tx_{n-3},Tx_{n-2},u) \\
&\quad ,d(Tx_{n-3},Tx_{n-2},u)\} \\
&\quad + \overline{c}[d(Tx_{n-3},Tx_{n-2},u)+d(Tx_{n-3},Tx_{n-2},u) \\
&\quad d(Tx_{n-3},Tx_{n-2},u)+d(Tx_{n-3},Tx_{n-2},u) \\
&\quad + d(Tx_{n-3},Tx_{n-1},Tx_n)] \\
&= [2(\overline{a}+\overline{b}+\overline{c})-\overline{b}]d(Tx_{n-3},Tx_{n-2},u) \\
&\le (2-\overline{b})d(Tx_{n-3},Tx_{n-2},u) \qquad (2.5)
\end{aligned}
$$

where $\overline{a}$, $\overline{b}$, $\overline{c}$ are evaluated at $(x_{n-2},x_n)$.

At the bottom line of the above inequality, $d(Tx_{n-3},Tx_{n-1},Tx_n)=0$. Because, let $d(Tx_{n-3},Tx_{n-1},Tx_n)\neq 0$, then applying (2.2), we get

$$
\begin{aligned}
d(Tx_{n-3},Tx_{n-1},Tx_n) &= d(Tx_{n-1},Tx_n,Tx_{n-3}) \\
&\le a\max\{d(fx_{n-1},Tx_{n-1},Tx_{n-3}),d(fx_n,Tx_n,Tx_{n-3})\} \\
&\quad + b\max\{d(fx_{n-1},Tx_{n-1},Tx_{n-3}),d(fx_n,Tx_n,Tx_{n-3})\} \\
&\quad + c[d(fx_{n-1},Tx_{n-1},Tx_{n-3})+d(fx_n,Tx_n,Tx_{n-3})] \\
&\le a\max\{d(Tx_{n-2},Tx_{n-1},Tx_{n-3}),d(Tx_{n-1},Tx_n,Tx_{n-3})\} \\
&\quad + b\max\{d(Tx_{n-2},Tx_{n-1},Tx_{n-3}),d(Tx_{n-1},Tx_n,Tx_{n-3})\} \\
&\quad + c[d(Tx_{n-2},Tx_{n-1},Tx_{n-3})+d(Tx_{n-1},Tx_n,Tx_{n-3})] \\
&= (a+b+c)d(Tx_{n-1},Tx_n,Tx_{n-3}) \\
&< d(Tx_{n-1},Tx_n,Tx_{n-3}) \qquad \blacksquare
\end{aligned}
$$

a contradiction. Thus, $d(Tx_{n-3},Tx_{n-1},Tx_n)=0$.

On using (2.3), (2.4),and (2.5), we get

$$
\begin{aligned}
d(Tx_{n-1},Tx_n,u) &= d(y_n,y_{n+1},u) \\
&\le ad(Tx_{n-2},Tx_{n-1},u)+bd(Tx_{n-2},Tx_{n-1},u) \\
&\quad + c[(2-\overline{b})d(Tx_{n-3},Tx_{n-2},u)] \\
&\le ad(Tx_{n-3},Tx_{n-2},u)+bd(Tx_{n-3},Tx_{n-2},u) \\
&\quad + c(2-\overline{b})d(Tx_{n-3},Tx_{n-2},u) \\
&= (a+b+2c)d(Tx_{n-3},Tx_{n-2},u)-\overline{b}cd(Tx_{n-3},Tx_{n-2},u) \\
&\le (1-\overline{b}c)d(Tx_{n-3},Tx_{n-2},u) \\
&\le (1-\beta\gamma)d(Tx_{n-3},Tx_{n-2},u) \\
&\le (1-\beta\gamma)^{\frac{n}{2}}d(y_0,y_1,u) \qquad (2.6)
\end{aligned}
$$

Hence $\{y_n\}$ is a Cauchy sequence.

For case (a) and (b), suppose that $X$ is complete.Then Cauchy sequence $\{y_n\}$ will converge to a point $p$ in $X$.

**Case (a):** Since $f$ is surjective, then there exist a point $z$ in $X$ such that $p = fz$. Now applying inequality (1.2), we get

$$\begin{aligned} d(fz,Tz,u) &\le d(fz,y_{n+1},u) + d(fz,Tz,y_{n+1}) + d(Tz,y_{n+1},u) \\ &\le d(fz,y_{n+1},u) + d(fz,Tz,y_{n+1}) + d(Tx_n,Tz,u) \\ &\le d(fz,y_{n+1},u) + d(fz,Tz,y_{n+1}) \\ &\quad + a(x,y)\max\{d(fx_n,fz,u),d(fz,Tz,u)\} \\ &\quad + b(x,y)\max\{d(fx_n,Tx_n,u),d(fz,Tz,u),d(fz,Tx_n,u)\} \\ &\quad + c(x,y)[d(fx_n,Tz,u)+d(fz,Tx_n,u)] \\ &\le \sup_{x,y\in X}[a(x,y)+c(x,y)]\max\big[\max\{d(fx_n,fz,u),d(fz,Tz,u)\} \\ &\quad ,d(fz,fx_{n+1},u)\big] + \sup_{x,y\in X}[b(x,y)+c(x,y)]\max\big[\max\{d(fx_n,fx_{n+1},u) \\ &\quad ,d(fz,Tz,u),d(fz,fx_{n+1},u)\},d(fx_n,Tz,u)+d(fz,fx_{n+1},u)\big] \\ &\quad + d(fz,y_{n+1},u) + d(fz,Tz,y_{n+1}) \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$d(fz,Tz,u) \le \sup x,y \in X(b+c)d(fz,Tz,u) < d(fz,Tz,u)$$

implies that $fz = Tz$.

**Case (b):** Since $f$ is continuous and $f$ and $T$ are compatible, we have

$$\lim_{n\to\infty} fy_n = fp \ \text{ and } \lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_{n+1} = p$$

and hence

$$\lim_{n\to\infty} d(fTx_n,Tfx_n,u) = 0 \tag{2.7}$$

Using (2.5), we get

$$\begin{aligned} d(fp,Tp,u) &\le d(fp,fy_{n+1},Tp) + d(fp,fy_{n+1},u) + d(fy_{n+1},u,Tp) \\ &\le d(fp,fy_{n+1},Tp) + d(fp,fy_{n+1},u) + d(Tp,Tfx_n,u) \\ &\le d(fp,fy_{n+1},Tp) + d(fp,fy_{n+1},u) \\ &\quad + a\max\{d(ffx_n,fp,u),d(fp,Tp,u)\} \\ &\quad + b\max\{d(ffx_n,Tfx_n,u),d(fp,Tp,u),d((fp,Tfx_n,u)\} \\ &\quad + c[d(ffx_n,Tp,u)+d(fp,Tfx_n,u)] \\ &\le d(fp,fy_{n+1},Tp) + d(fp,fy_{n+1},u) \\ &\quad + \sup_{x,y\in X}[a(x,y)+b(x,y)+c(x,y)]\max\Big\{\max\{d(ffx_n,fp,u), \\ &\quad d(fp,Tp,u)\},\max\{d(ffx_n,Tfx_n,u),d(fp,Tp,u), \\ &\quad d((fp,Tfx_n,u)\},c[d(ffx_n,Tp,u)+d(fp,Tfx_n,u)]\Big\} \end{aligned} \tag{2.8}$$

Now we have

$$d(ffx_n,Tfx_n,u) \le d(ffx_n,fTx_n,u) + d(fTx_n,Tfx_n,u) + d(ffx_n,Tfx_n,Tfx_n)$$

Using the continuity of $f$ and the compatibility of $f$ and $T$, it follows that

$$\lim_{n\to\infty} d(ffx_n,Tfx_n,u) = 0, \ \lim_{n\to\infty} d(ffx_n,fTx_n,u) = 0 \tag{2.9}$$

$$\lim_{n\to\infty} ffx_n = fp, \text{ impliesthat, } \lim_{n\to\infty} Tfx_n = fp$$

Taking limit as $n \to \infty$ and using the inequality (2.7) and (2.8), we get

$$d(fp, Tp, u) \le \sup_{x,y\in X}[a(x,y)+b(x,y)+c(x,y)]d(fp, Tp, u), \text{ , impliesthat, , } fp = Tp$$

**Case (c):**In this case, $p \in f(X)$. Let $z \in f^{-1}p$, then $p = fz$, and the proof is completed by Case (a).
To establish uniqueness, suppose that $q$ is another coincidence point of $f$ and $T$. Then from (1.2) with $a,\ b,\ c$ evaluated at $(p, q)$, we have

$$\begin{aligned} d(Tp, Tq, u) &\le a \max\{d(fp, fq, u), d(fq, Tq, u)\} \\ &\quad + b \max\{d(fp, Tp, u), d(fq, Tq, u), d(fq, Tp, u)\} \\ &\quad + c[d(fp, Tq, u) + d(fq, Tp, u)] \\ &\le (a + b + 2c)d(Tp, Tq, u) \end{aligned}$$

Hence $Tp = Tq$.

COROLLARY 2.1. *Let $(X, d)$ be a complete 2-metric space and $T$ be a self map of $X$ satisfying* (1.2) *with $f = I$, the identity mapping on $X$. Then $T$ has a unique fixed point and at this fixed point $T$ is continuous.*

PROOF. The existence and uniqueness of the fixed point comes from Theorem (2.1) by setting $f = I$.
To prove continuity at the unique fixed point $p$, we apply inequality (1.2), where $a,\ b,\ c$ are evaluated at $(y_n, p)$.

$$\begin{aligned} d(Ty_n, p, u) &= d(Ty_n, Tp, u) \\ &\le a \max\{d(y_n, p, u), d(p, Tp, u)\} \\ &\quad + b \max\{d(y_n, Ty_n, u), d(p, Tp, u), d(p, Ty_n, u)\} \\ &\quad + c[d(y_n, Tp, u) + d(p, Ty_n, u)] \end{aligned}$$

Taking limit as $n \to \infty$ yields

$$\lim_{n\to\infty} d(Ty_n, p, u) \le (b + c) \lim_{n\to\infty} d(p, Ty_n, u) < \lim_{n\to\infty} d(p, Ty_n, u)$$

a contradiction. Therefore, $\lim_{n\to\infty} Ty_n = p = Tp$. □

REMARK 2.1. Our condition (1.2) includes condition (1.1) of [7] if we define, with $f = I$ the identity mapping,

$$m(x, y, u) = \max\{d(x, y, u), d(x, Tx, u), d(y, Ty, u), \frac{1}{2}[d(x, Ty, u) + d(y, Tx, u)]\}.$$

For each $x, y \in X$ such that

$$m(x, y, u) = \max\{d(x, Tx, u), d(y, Ty, u)\}$$

define $a(x, y) = 0,\ b(x, y) = a + b,\ c(x, y) = c$.
For each $x, y \in X$ such that

$$m(x, y, u) = \frac{1}{2}[d(x, Ty, u) + d(y, Tx, u)]$$

define $a(x, y) = 0,\ b(x, y) = b,\ c(x, y) = a + 2c$. Hence our Theorem (2.1) is a proper generalization of [7].

□

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