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ω-OPEN SETS AND DECOMPOSITIONS OF CONTINUITY

[1]Otchana Thevar Ravi, [2]Malliharjunaiah Paranjothi,
[3]Ilangovan Rajasekaran and [4]Soundararajan Satheesh Kanna

Abstract. In this paper, we introduce some generalized classes of τ_ω in topological spaces and investigate their properties. Also we study the relations between the new generalized classes of τ_ω and the other established generalized classes of τ_ω. With the help of all such classes, we obtain some new decompositions of continuity.

1. Introduction

In 1982, Hdeib [**2**] introduced ω-closed sets and ω-open sets in topological spaces. In 2009, Noiri et al [**3**] introduced some weaker forms of ω-open sets and obtained some decompositions of continuity. Quite Recently, Ravi et al [**4**] introduced the notion of semi-ω-open sets in topological spaces. Further they studied the weaker and stronger forms of such classes. The properties of such classes were also studied in detail.

In this paper, further study is carried out using the established generalized classes of τ_ω. Apart from this, some new generalizations of ω-open sets are introduced and investigated. Also, using all the generalized subsets of τ_ω, some new decompositions of continuity in topological spaces are obtained.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Q}^\star$, $\mathbb{Z}$) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers, the set of all integers).

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By a space (X, τ), we always mean a topological space (X, τ) with no separation properties assumed. If H ⊂ X, cl(H) and int(H) will, respectively, denote the closure and interior of H in (X, τ). τ_u denotes the usual topology on $\mathbb{R}$.

Definition 2.1. ([**5**]) Let H be a subset of a space (X, τ). A point p in X is called a condensation point of H if for each open set U containing p, $U \cap H$ is uncountable.

Definition 2.2. ([**2**]) A subset H of a space (X, τ) is called ω-closed if it contains all its condensation points.
The complement of an ω-closed set is called ω-open.

It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Definition 2.3. ([**3**]) A subset H of a space (X, τ) is called

(1) α-ω-open if $\mathrm{H} \subset \mathrm{int}_\omega(\mathrm{cl}(\mathrm{int}_\omega(\mathrm{H})))$;
(2) pre-ω-open if $\mathrm{H} \subset \mathrm{int}_\omega(\mathrm{cl}(\mathrm{H}))$;
(3) β-ω-open if $\mathrm{H} \subset \mathrm{cl}(\mathrm{int}_\omega(\mathrm{cl}(\mathrm{H})))$;
(4) b-ω-open if $\mathrm{H} \subset \mathrm{int}_\omega(\mathrm{cl}(\mathrm{H})) \cup \mathrm{cl}(\mathrm{int}_\omega(\mathrm{H}))$.

Definition 2.4. ([**4**]) A subset H of a space (X, τ) is called semi-ω-open if $H \subset cl(int_\omega(H))$.

Remark 2.1. ([**3, 4**]) The diagram holds for any subset of a space (X, τ):

$$\begin{array}{ccccccc} \textbf{open} & \longrightarrow & \alpha\text{-}\omega\textbf{-open} & \longrightarrow & \textbf{pre-}\omega\textbf{-open} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \omega\textbf{-open} & \longrightarrow & \textbf{semi-}\omega\textbf{-open} & \longrightarrow & \textbf{b-}\omega\textbf{-open} & \longrightarrow & \beta\text{-}\omega\textbf{-open} \end{array}$$

In this diagram, none of the implications is reversible.

Definition 2.5. ([**3**]) A subset H of a space (X, τ) is called

(1) an ω-t-set if $\mathrm{int(H)} = \mathrm{int}_\omega(\mathrm{cl}(\mathrm{H}))$;
(2) an ω-$\mathcal{B}$-set if H= U ∩ V, where $\mathrm{U} \in \tau$ and V is an ω-t-set.

Definition 2.6. ([**4**]) A subset H of a space (X, τ) is called an $\omega^\star$-t-set if $int_\omega(cl(H)) = int_\omega(H)$.

Lemma 2.1 ([**1**]). *If U is an open set, then $cl(U \cap H) = cl(U \cap cl(H))$ and hence $U \cap cl(H) \subset cl(U \cap H)$ for any subset H.*

3. On new subsets of τ_ω

Definition 3.1. A subset H of a space (X, τ) is called

(1) an $\omega^{\#}$-t-set if $int(H) = cl(int_\omega(H))$;
(2) an $\omega^{\#}$-$\mathcal{B}$-set if $H = U \cap V$, where $U \in \tau$ and V is an $\omega^{\#}$-t-set.

REMARK 3.1. (1) In $\mathbb{R}$ with usual topology τ_u, a subset H with $int(H) = \phi$ is an $\omega^{\#}$-t-set if and only if $int(H) = \phi = int_\omega(H)$.

(2) In $\mathbb{R}$ with usual topology τ_u, there is no proper subset H, with $int(H) \neq \phi$ which is an $\omega^{\#}$-t-set. (or) The only subset in $\mathbb{R}$, with nonempty interior, which is an $\omega^{\#}$-t-set is $\mathbb{R}$ itself.

PROOF. (1). Proof is direct from the definition.

(2). If H is a proper $\omega^{\#}$-t-set in $\mathbb{R}$ with $int(H) \neq \phi$, then

$$int(H) = cl(int_\omega(H)) = cl(int(H)).$$

Thus int(H) is a proper clopen subset in $\mathbb{R}$ contradicting the connectedness of $\mathbb{R}$. Hence the result. □

EXAMPLE 3.1. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = [0, 1] \cap \mathbb{Q}$ is an $\omega^{\#}$-t-set by (1) of Remark 3.1 since $int(H) = \phi = int_\omega(H)$.
(2) $H = \mathbb{Q}^\star$ is not an $\omega^{\#}$-t-set by (1) of Remark 3.1 since $int(H) = \phi \neq int_\omega(H)$.
(3) $H = (0, 1]$ is not an $\omega^{\#}$-t-set by (2) of Remark 3.1 since $int(H) \neq \phi$.

REMARK 3.2. In a space (X, τ),

(1) Every open set is an $\omega^{\#}$-$\mathcal{B}$-set.
(2) Every $\omega^{\#}$-t-set is an $\omega^{\#}$-$\mathcal{B}$-set.

EXAMPLE 3.2. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = [0, 1] \cap \mathbb{Q}$ is an $\omega^{\#}$-$\mathcal{B}$-set by (2) of Remark 3.2 since H is an $\omega^{\#}$-t-set by (1) of Example 3.1.
(2) $H = \mathbb{Q}^\star$ is not an $\omega^{\#}$-$\mathcal{B}$-set. If $H = U \cap V$ where U is open and V is an $\omega^{\#}$-t-set, then $H \subset U$ and $H \subset V$. Since $\mathbb{R}$ is the only open set containing H, $U = \mathbb{R}$ and thus $H = \mathbb{R} \cap V = V$. This is a contradiction since H is not an $\omega^{\#}$-t-set by (2) of Example 3.1.

REMARK 3.3. The converses of (1) and (2) in Remark 3.2 are not true as seen from the following Example.

EXAMPLE 3.3. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = [0, 1] \cap \mathbb{Q}$ is an $\omega^{\#}$-$\mathcal{B}$-set by (1) of Example 3.2. But H is not open.
(2) $H = (0, 1)$ is an $\omega^{\#}$-$\mathcal{B}$-set by (1) of Remark 3.2. But H is not an $\omega^{\#}$-t-set by (2) of Remark 3.1.

PROPOSITION 3.1. *If A and B are $\omega^{\#}$-t-sets of a space (X, τ), then $A \cap B$ is an $\omega^{\#}$-t-set.*

PROOF. Let A and B be $\omega^{\#}$-t-sets. Then we have $int(A \cap B) \subset int_\omega(A \cap B) \subset cl(int_\omega(A \cap B)) = cl(int_\omega(A) \cap int_\omega(B)) \subset cl(int_\omega(A)) \cap cl(int_\omega(B)) = int(A) \cap int(B) = int(A \cap B)$. Thus $int(A \cap B) = cl(int_\omega(A \cap B))$ and hence $A \cap B$ is an $\omega^{\#}$-t-set. □

THEOREM 3.1. *For a subset H of a space (X, τ), the following are equivalent:*

(1) *H is open;*
(2) *H is semi-ω-open and an $\omega^{\#}$-$\mathcal{B}$-set.*

PROOF. (1)⇒(2): (2) follows by Remark 2.1 and (1) of Remark 3.2.
(2)⇒(1): Given H is semi-ω-open and an $\omega^{\#}$-$\mathcal{B}$-set. So $H = U \cap V$ where U is open and $int(V) = cl(int_\omega(V))$. Then $H \subset U = int(U)$. Also H is semi-ω-open implies $H \subset cl(int_\omega(H)) \subset cl(int_\omega(V)) = int(V)$ by assumption. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open. □

REMARK 3.4. The concepts of semi-ω-openness and being an $\omega^{\#}$-$\mathcal{B}$-set are independent.

EXAMPLE 3.4. In $\mathbb{R}$ with usual topology τ_u,
(1) $H = \mathbb{Q}^\star$ is semi-ω-open but not an $\omega^{\#}$-$\mathcal{B}$-set, by (2) of Example 3.2.
(2) $H = [0,1] \cap \mathbb{Q}$ is an $\omega^{\#}$-$\mathcal{B}$-set by (1) of Example 3.2. But H is not semi-ω-open.

DEFINITION 3.2. A subset H of a space (X, τ) is called
(1) an $\omega^{\#}$-t_α-set if $int(H) = cl(int_\omega(cl(H)))$;
(2) an $\omega^{\#}$-$\mathcal{B}_\alpha$-set if $H = U \cap V$, where $U \in \tau$ and V is an $\omega^{\#}$-t_α-set.

EXAMPLE 3.5. (1) In $\mathbb{R}$ with usual topology τ_u, $\mathbb{Z}$ is an $\omega^{\#}$-t$_\alpha$-set since $int(\mathbb{Z}) = \phi$ and $cl(int_\omega(cl(\mathbb{Z}))) = cl(int_\omega(\mathbb{Z})) = cl(\phi) = \phi$.
(2) In $\mathbb{R}$ with usual topology τ_u, $H = (0,1)$ is not an $\omega^{\#}$-t_α-set since $int(H) = (0,1)$, but $cl(int_\omega(cl(H))) = cl(int_\omega([0,1])) = cl((0,1)) = [0,1]$.
(3) In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}^\star$ is not an $\omega^{\#}$-t_α-set since $int(\mathbb{Q}^\star) = \phi$, but $cl(int_\omega(cl(\mathbb{Q}^\star))) = cl(int_\omega(\mathbb{R})) = cl(\mathbb{R}) = \mathbb{R}$.

REMARK 3.5. In a space (X, τ),
(1) Every open set is $\omega^{\#}$-$\mathcal{B}_\alpha$-set.
(2) Every $\omega^{\#}$-t_α-set is $\omega^{\#}$-$\mathcal{B}_\alpha$-set.

EXAMPLE 3.6. (1) In $\mathbb{R}$ with usual topology τ_u, $\mathbb{Z}$ is an $\omega^{\#}$-t_α-set by (1) of Example 3.5 and hence an $\omega^{\#}$-$\mathcal{B}_\alpha$-set by (2) of Remark 3.5.
(2) In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}^\star$ is not an $\omega^{\#}$-$\mathcal{B}_\alpha$-set. If $H = U \cap V$ where U is open and V is an $\omega^{\#}$-t_α-set, then $H \subset U$ and $H \subset V$. But $\mathbb{R}$ is the only open set containing H. Hence $U = \mathbb{R}$ and $H = \mathbb{R} \cap V = V$ which means that H is an $\omega^{\#}$-t_α-set which is a contradiction by (3) of Example 3.5. Thus H is not an $\omega^{\#}$-$\mathcal{B}_\alpha$-set.

REMARK 3.6. The converses of (1) and (2) in Remark 3.5 are not true as seen from the following Example.

EXAMPLE 3.7. (1) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is an $\omega^{\#}$-$\mathcal{B}_\alpha$-set by (2) of Remark 3.5 since H is an $\omega^{\#}$-t_α-set. But H is not open.
(2) In $\mathbb{R}$ with usual topology τ_u, $H = (0,1)$ is an $\omega^{\#}$-$\mathcal{B}_\alpha$-set by (1) of Remark 3.5 since H is open but not an $\omega^{\#}$-t_α-set by (2) of Example 3.5.

PROPOSITION 3.2. *If A and B are $\omega^{\#}$-t_α-sets of a space (X,τ), then $A \cap B$ is an $\omega^{\#}$-t_α-set.*

PROOF. Let A and B be $\omega^{\#}$-t_α-sets. Then we have $int(A \cap B) \subset int_\omega(A \cap B) \subset int_\omega(cl(A \cap B)) \subset cl(int_\omega(cl(A \cap B))) \subset cl(int_\omega(cl(A))) \cap cl(int_\omega(cl(B))) = int(A) \cap int(B) = int(A \cap B)$. Then $int(A \cap B) = cl(int_\omega(cl(A \cap B)))$ and hence $A \cap B$ is an $\omega^{\#}$-t_α-set.

The following Examples show that β-ω-openness and being an $\omega^{\#}$-$\mathcal{B}_\alpha$-set are independent. □

EXAMPLE 3.8. (1) In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}^\star$ is not an $\omega^{\#}$-$\mathcal{B}_\alpha$-set, by (2) of Example 3.6. But it is β-ω-open.

(2) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is an $\omega^{\#}$-$\mathcal{B}_\alpha$-set by (1) of Example 3.7. But H is not β-ω-open.

THEOREM 3.2. *For a subset H of a space (X,τ), the following are equivalent:*

(1) *H is open;*
(2) *H is β-ω-open and an $\omega^{\#}$-$\mathcal{B}_\alpha$-set.*

PROOF. (1)⇒(2): follows by Remark 2.1 and (1) of Remark 3.5.

(2)⇒(1): Given H is β-ω-open and an $\omega^{\#}$-$\mathcal{B}_\alpha$-set. So $H = U \cap V$ where U is open and V is an $\omega^{\#}$-t_α-set. Then $H \subset U = int(U)$. Also H is β-ω-open implies $H \subset cl(int_\omega(cl(H))) \subset cl(int_\omega(cl(V))) = int(V)$ since V is an $\omega^{\#}$-t_α-set. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open. □

DEFINITION 3.3. A subset H of a space (X,τ) is called a strong ω-$\mathcal{B}$-set if $H = U \cap V$, where $U \in \tau$ and V is a ω-t-set and $int_\omega(cl(V)) = cl(int_\omega(V))$.

EXAMPLE 3.9. (1) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is a strong ω-$\mathcal{B}$-set. Since $int(H) = int_\omega(cl(H)) = cl(int_\omega(H)) = \phi$, H is a ω-t-set with $int_\omega(cl(H)) = cl(int_\omega(H))$ and $H = \mathbb{R} \cap H$, H is a strong ω-$\mathcal{B}$-set.

(2) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}^\star, \mathbb{Q}^\star \cup \mathbb{N}\}$, $H = \mathbb{Q}$ is not a strong ω-$\mathcal{B}$-set. If $H = U \cap V$ where U is open and V is an ω-t-set with $int_\omega(cl(H)) = cl(int_\omega(H))$ then we have $H \subset U$ and $\mathbb{R}$ is the only open set containing H. Hence $U = \mathbb{R}$ and $H = \mathbb{R} \cap V = V$. But this is a contradiction since $int_\omega(cl(H)) = int_\omega(\mathbb{Q}) = \mathbb{N} \neq \mathbb{Q} = cl(int_\omega(H))$. Thus H is not a strong ω-$\mathcal{B}$-set.

PROPOSITION 3.3. *In a space (X,τ),*

(1) *Every open set is a strong ω-$\mathcal{B}$-set.*
(2) *Every ω-t-set H with $int_\omega(cl(H)) = cl(int_\omega(H))$ is a strong ω-$\mathcal{B}$-set.*

PROOF. Proof follows directly from the definition of a strong ω-$\mathcal{B}$-set. □

REMARK 3.7. The converses of (1) and (2) in Proposition 3.3 are not true as seen from the following Example.

EXAMPLE 3.10. (1) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is a strong ω-$\mathcal{B}$-set by (1) of Example 3.9. But H is not open.

(2) In $\mathbb{R}$ with usual topology τ_u, $H = (0,1)$ is open and hence by (1) of Proposition 3.3, H is a strong ω-$\mathcal{B}$-set. But $int_\omega(cl(H)) = (0,1) \neq [0,1] = cl(int_\omega(H))$ and hence H is not a ω-t-set with $int_\omega(cl(H)) = cl(int_\omega(H))$.

PROPOSITION 3.4. *In a space (X,τ), every strong ω-$\mathcal{B}$-set is a ω-$\mathcal{B}$-set.*

PROOF. Proof follows from the fact that an ω-t-set H with $int_\omega(cl(H)) = cl(int_\omega(H))$ is an ω-t-set. □

REMARK 3.8. The converse of Proposition 3.4 is not true as seen from the following Example.

EXAMPLE 3.11. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}^\star, \mathbb{Q}^\star \cup \mathbb{N}\}$, $H = \mathbb{Q}$ is an ω-$\mathcal{B}$-set since H is an ω-t-set. But H is not a strong ω-$\mathcal{B}$-set by (2) of Example 3.9.

THEOREM 3.3. *For a subset H of a space (X,τ), the following are equivalent:*

(1) *H is open;*
(2) *H is b-ω-open and a strong ω-$\mathcal{B}$-set.*

PROOF. (1)⇒(2): (2) follows by Remark 2.1 and Proposition 3.3.

(2)⇒(1): Given H is b-ω-open and a strong ω-$\mathcal{B}$-set. Since H is a strong ω-$\mathcal{B}$-set, $H = U \cap V$ where U is open and V is an ω-t-set with $int_\omega(cl(V)) = cl(int_\omega(V))$. Then $H \subset U = int(U)$. Also H is b-ω-open implies $H \subset int_\omega(cl(H)) \cup cl(int_\omega(H)) \subset int_\omega(cl(V)) \cup cl(int_\omega(V)) = int(V)$ by assumption. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open. □

REMARK 3.9. The concepts of b-ω-openness and being a strong ω-$\mathcal{B}$-set are independent.

EXAMPLE 3.12. (1) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is a strong-ω-$\mathcal{B}$-set by (1) of Example 3.9. But H is not b-ω-open.
(2) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}^\star, \mathbb{Q}^\star \cup \mathbb{N}\}$, $H = \mathbb{Q}$ is not a strong-ω-$\mathcal{B}$-set by (2) of Example 3.9. But H is b-ω-open.

4. Semi-$\omega^\star$-regular, pre-$\omega^\star$-regular and β-$\omega^\star$-regular sets

DEFINITION 4.1. A subset H of a space (X,τ) is called semi-$\omega^\star$-regular if H is semi-ω-open and an ω-t-set.

EXAMPLE 4.1. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = (0,1]$ is semi-ω-open as well as an ω-t-set and hence is semi-$\omega^\star$-regular.
(2) $H = \mathbb{Q}$ is not semi-ω-open and hence not semi-$\omega^\star$-regular.

REMARK 4.1. In a space (X,τ),

(1) Every semi-$\omega^\star$-regular set is semi-ω-open.
(2) Every semi-$\omega^\star$-regular set is an ω-t-set.

REMARK 4.2. The converses of (1) and (2) in Remark 4.1 are not true as seen from the following Example.

EXAMPLE 4.2. (1) In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}^\star$ is semi-ω-open but H is not semi-$\omega^\star$-regular because it is not an ω-t-set.
(2) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is an ω-t-set. But H is not semi-$\omega^\star$-regular because it is not semi-ω-open.

DEFINITION 4.2. A subset H of a space (X, τ) is called an ω-$\mathcal{SB}$-set if $H = U \cap V$ where $U \in \tau$ and V is a semi-$\omega^\star$-regular.

EXAMPLE 4.3. In $\mathbb{R}$ with usual topology τ_u,
(1) $H = (0, 1]$ is semi-$\omega^\star$-regular by (1) of Example 4.1 and hence an ω-$\mathcal{SB}$-set for $H = \mathbb{R} \cap H$.
(2) $H = \mathbb{Q}^\star$ is not an ω-$\mathcal{SB}$-set. If $H = U \cap V$ where U is open and V is semi-$\omega^\star$-regular, then $H \subset U$. Since $\mathbb{R}$ is the only open set containing H, $U = \mathbb{R}$ and so $H = \mathbb{R} \cap V = V$. This means that H is semi-$\omega^\star$-regular which is a contradiction by (1) of Example 4.2.

PROPOSITION 4.1. *In a space (X, τ),*
(1) *Every open set is an ω-$\mathcal{SB}$-set.*
(2) *Every semi-$\omega^\star$-regular set is an ω-$\mathcal{SB}$-set.*

PROOF. This is obvious from the definition of an ω-$\mathcal{SB}$-set. □

REMARK 4.3. The converses of (1) and (2) in Proposition 4.1 are not true as seen from the following Example.

EXAMPLE 4.4. (1) In $\mathbb{R}$ with usual topology τ_u, $H = (0, 1]$ is semi-$\omega^\star$-regular by (1) of Example 4.1 and hence is an ω-$\mathcal{SB}$-set. But H is not open.
(2) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}^\star$ is open and hence an ω-$\mathcal{SB}$-set. But H is not semi-$\omega^\star$-regular by (2) of Example 4.2.

THEOREM 4.1. *For a subset H of a space (X, τ), the following are equivalent:*
(1) *H is open;*
(2) *H is pre-ω-open and an ω-$\mathcal{SB}$-set.*

PROOF. . (1)⇒(2): This is obvious by Remark 2.1 and (1) of Proposition 4.1.
(2)⇒(1): Given H is pre-ω-open and an ω-$\mathcal{B}$-set. Since H is an ω-$\mathcal{SB}$-set, $H = U \cap V$ where U is open and V is semi-$\omega^\star$-regular. Then we have $H \subset U = int(U)$. Also H is pre-ω-open implies $H \subset int_\omega(cl(H)) \subset int_\omega(cl(V)) = int(V)$ for V is an ω-t-set being semi-$\omega^\star$-regular. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open. □

REMARK 4.4. The notions of pre-ω-openness and being an ω-$\mathcal{SB}$-set are independent.

EXAMPLE 4.5. In $\mathbb{R}$ with usual topology τ_u,
(1) $H = (0, 1]$ is an ω-$\mathcal{SB}$-set by (1) of Example 4.3 but not pre-ω-open.
(2) $H = \mathbb{Q}^\star$ is pre-ω-open but not an ω-$\mathcal{SB}$-set by (2) of Example 4.3.

Definition 4.3. A subset H of a space (X, τ) is called pre-$\omega^\star$-regular if H is pre-ω-open and an $\omega^\#$-t-set.

Example 4.6. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = \mathbb{Q}$ is an $\omega^\#$-t-set and also it is pre-ω-open. Hence it is pre-$\omega^\star$-regular.
(2) $H = \{1\}$ is not pre-$\omega^\star$-regular because it is not pre-ω-open.

Proposition 4.2. *In a space (X, τ),*

(1) *Every pre-$\omega^\star$-regular set is a pre-ω-open.*
(2) *Every pre-$\omega^\star$-regular set is an $\omega^\#$-t-set.*

Remark 4.5. The converses of (1) and (2) in Proposition 4.2 are not true as seen from the following Example.

Example 4.7. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = (0, 1)$ is pre-ω-open. But H is not pre-$\omega^\star$-regular because it is not an $\omega^\#$-t-set by (2) of Remark 3.1.
(2) $H = \{1\}$ is an $\omega^\#$-t-set but not pre-$\omega^\star$-regular by (2) of Example 4.6.

Definition 4.4. A subset H of a space (X, τ) is called an ω-$\mathcal{SC}$-set if $H = U \cap V$ where $U \in \tau$ and V is pre-$\omega^\star$-regular.

Example 4.8. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = \mathbb{Q}$ is pre-$\omega^\star$-regular by (1) of Example 4.6 and hence is an ω-$\mathcal{SC}$-set.
(2) $H = \mathbb{Q}^\star$ is not an ω-$\mathcal{SC}$-set. If $H = U \cap V$ where U is open and V is pre-$\omega^\star$-regular, then we have $H \subset U$ and $H \subset V$. Since $\mathbb{R}$ is the only open set containing H, $U = \mathbb{R}$ and so $H = \mathbb{R} \cap V = V$ which is pre-$\omega^\star$-regular. This is a contradiction since H is not an $\omega^\#$-t-set by (1) of Remark 3.1 and hence not pre-$\omega^\star$-regular.

Proposition 4.3. *In a space (X, τ),*

(1) *Every open set is an ω-$\mathcal{SC}$-set.*
(2) *Every pre-$\omega^\star$-regular set is an ω-$\mathcal{SC}$-set.*

Proof. Proof follows directly from the definition of an ω-$\mathcal{SC}$-set. □

Remark 4.6. The converses of (1) and (2) in Proposition 4.3 are not true as seen from the following Example.

Example 4.9. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = \mathbb{Q}$ is ω-$\mathcal{SC}$-set by (1) of Example 4.8. But H is not open.
(2) $H = (0, 1)$ is open and hence ω-$\mathcal{SC}$-set but not pre-$\omega^\star$-regular by (1) of Example 4.7.

Theorem 4.2. *For a subset H of a space (X, τ), the following are equivalent:*

(1) *H is open;*
(2) *H is semi-ω-open and an ω-$\mathcal{SC}$-set.*

PROOF. . (1)⇒(2): (2) follows from Remark 2.1 and (1) of Proposition 4.3.

(2)⇒(1): Given H is semi-ω-open and an ω-$\mathcal{SC}$-set. Since H is an ω-$\mathcal{SC}$-set, $H = U \cap V$ where U is open and V is pre-$\omega^\star$-regular. Then $H \subset U = int(U)$ and $H \subset V$. Also H is semi-ω-open implies $H \subset cl(int_\omega(H)) \subset cl(int_\omega(V)) = int(V)$ since V is an $\omega^{\#}$-t-set being pre-$\omega^\star$-regular. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open. □

REMARK 4.7. The notions of semi-ω-openness and being an ω-$\mathcal{SC}$-set are independent.

EXAMPLE 4.10. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = \mathbb{Q}^\star$ is semi-ω-open but not an ω-$\mathcal{SC}$-set by (2) of Example 4.8.
(2) $H = \mathbb{Q}$ is ω-$\mathcal{SC}$-set by (1) of Example 4.8 but not semi-ω-open.

DEFINITION 4.5. A subset H of a space (X, τ) is called

(1) an ω-t_α-set [**3**] if $int(H) = int_\omega(cl(int_\omega(H)))$.
(2) β-$\omega^\star$-regular if H is β-ω-open and an ω-t_α-set.

EXAMPLE 4.11. (1) In $\mathbb{R}$ with usual topology τ_u, $H = (0, 1]$ is β-ω-open as well as ω-t_α-set and hence is β-$\omega^\star$-regular.
(2) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is not β-ω-open and hence is not β-$\omega^\star$-regular.
(3) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}^\star$ is not ω-t_α-set and hence is not β-$\omega^\star$-regular.

PROPOSITION 4.4. *In a space (X, τ), every ω-t-set is an ω-t_α-set.*

PROOF. Let H be an ω-t-set. The $int(H) = int_\omega(cl(H))$. Now

$$int_\omega(cl(int_\omega(H))) \subset int_\omega(cl(H)) = int(H).$$

We have $int(H) \subset int_\omega(H) \subset int_\omega(cl(int_\omega(H)))$. Thus we obtain that $int(H) = int_\omega(cl(int_\omega(H)))$ and so H is an ω-t_α-set. □

EXAMPLE 4.12. In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}$ is an ω-t_α-set but not an ω-t-set.

PROPOSITION 4.5. *In a space (X, τ), every $\omega^{\#}$-t-set is an ω-t_α-set.*

PROOF. Let H be an $\omega^{\#}$-t-set. Then $int(H) = cl(int_\omega(H))$. Now

$$int_\omega(cl(int_\omega(H))) \subset cl(int_\omega(H)) = int(H).$$

Also, $int(H) \subset int_\omega(H) \subset int_\omega(cl(int_\omega(H)))$. Thus we obtain that $int(H) = int_\omega(cl(int_\omega(H)))$ and so H is an ω-t_α-set. □

EXAMPLE 4.13. In $\mathbb{R}$ with usual topology τ_u, $H = (0, 1]$ is an ω-t_α-set but not an $\omega^{\#}$-t-set.

PROPOSITION 4.6. *In a space (X, τ),*

(1) *Every semi-$\omega^\star$-regular set is β-$\omega^\star$-regular.*
(2) *Every pre-$\omega^\star$-regular set is β-$\omega^\star$-regular.*

(3) *Every β-$\omega^\star$-regular set is a β-ω-open.*
(4) *Every β-$\omega^\star$-regular set is an ω-t_α-set.*

PROOF. (1). Proof follows from the fact that every semi-ω-open set is β-ω-open and every ω-t-set is an ω-t_α-set.

(2). Proof follows from the fact that every pre-ω-open set is β-ω-open and every $\omega^\#$-t-set is an ω-t_α-set.

(3), (4). The proofs follow from their definitions. □

REMARK 4.8. The converses of (1), (2), (3) and (4) in Proposition 4.6 are not true as seen from the following Example.

EXAMPLE 4.14. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = \mathbb{Q}$ is an ω-t_α-set by Example 4.12 and also β-ω-open. Hence it is β-$\omega^\star$-regular. But H is not semi-ω-open and hence is not semi-$\omega^\star$-regular.
(2) $H = (0, 1]$ is β-$\omega^\star$-regular by (1) of Example 4.11. But H is not pre-ω-open and hence not pre-$\omega^\star$-regular.
(3) $H = \mathbb{Q}^\star$ is β-ω-open. But H is not an ω-t_α-set and hence is not β-$\omega^\star$-regular.
(4) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}$ is an ω-t_α-set. But H is not β-ω-open and hence is not β-$\omega^\star$-regular.

DEFINITION 4.6. A subset H of a space (X, τ) is called a weak ω-$\mathcal{AB}$-set if $H = U \cap V$, where $U \in \tau$ and V is β-$\omega^\star$-regular.

EXAMPLE 4.15. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = (0, 1]$ is β-$\omega^\star$-regular by (1) of Example 4.11 and hence a weak ω-$\mathcal{AB}$-set for $H = \mathbb{R} \cap H$.
(2) $H = \mathbb{Q}^\star$ is not a weak ω-$\mathcal{AB}$-set. If $H = U \cap V$ where U is open and V is β-$\omega^\star$-regular, then $H \subset U$. Since $\mathbb{R}$ is the only open set containing H, $U = \mathbb{R}$. So $H = \mathbb{R} \cap V = V$ and thus H is β-$\omega^\star$-regular. This is a contradiction since H is not β-$\omega^\star$-regular by (3) of Example 4.14. Thus $H = \mathbb{Q}^\star$ is not a weak ω-$\mathcal{AB}$-set.

PROPOSITION 4.7. *In a space (X, τ), every weak ω-$\mathcal{AB}$-set is β-ω-open.*

PROOF. Let H be a weak ω-$\mathcal{AB}$-set. Then $H = U \cap V$, where U is open and V is β-$\omega^\star$-regular. Hence V is β-ω-open. So $H = U \cap V \subset U \cap cl(int_\omega(cl(V))) \subset cl(U \cap int_\omega(cl(V))) = cl(int_\omega(U) \cap int_\omega(cl(V))) = cl(int_\omega(U \cap cl(V))) \subseteq cl(int_\omega(cl(U \cap V))) = cl(int_\omega(cl(H)))$ by Lemma 2.1. Hence H is β-ω-open. □

REMARK 4.9. The converse of Proposition 4.7 is not true as seen from the following Example.

EXAMPLE 4.16. In $\mathbb{R}$ with usual topology τ_u, $H = \mathbb{Q}^\star$ is β-ω-open. But H is not a weak ω-$\mathcal{AB}$-set by (2) of Example 4.15.

PROPOSITION 4.8. *In a space (X, τ),*

(1) *Every open set is a weak ω-$\mathcal{AB}$-set.*

(2) *Every β-$\omega^\star$-regular set is a weak ω-$\mathcal{AB}$-set.*

REMARK 4.10. The converses of (1) and (2) in Proposition 4.8 are not true as seen from the following Example.

EXAMPLE 4.17. (1) In $\mathbb{R}$ with usual topology τ_u, $H = (0, 1]$ is a weak ω-$\mathcal{AB}$-set by (1) of Example 4.15. But H is not open.

(2) In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$, $H = \mathbb{Q}^\star$ is open and hence a weak ω-$\mathcal{AB}$-set. But H is not β-$\omega^\star$-regular by (3) of Example 4.11.

THEOREM 4.3. *For a subset H of a space (X, τ), the following are equivalent:*

(1) *H is β-$\omega^\star$-regular.*

(2) *H is an ω-t_α-set and a weak ω-$\mathcal{AB}$-set.*

PROOF. (1)⇒(2): Proof follows directly since every β-$\omega^\star$-regular set is an ω-t_α-set by definition and a weak ω-$\mathcal{AB}$-set by (2) of Proposition 4.8.

(2)⇒(1): Let H be an ω-t_α-set and a weak ω-$\mathcal{AB}$-set. Since H is a weak ω-$\mathcal{AB}$-set, by Proposition 4.7 H is β-ω-open. Thus H is an ω-t_α-set as well as β-ω-open. Hence H is β-$\omega^\star$-regular. □

REMARK 4.11. The concepts of being an ω-t_α-set and being a weak ω-$\mathcal{AB}$-set are independent.

EXAMPLE 4.18. In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^\star\}$,

(1) $H = \mathbb{Q}$ is an ω-t_α-set by (4) of Example 4.14. If H is a weak ω-$\mathcal{AB}$-set, then $\mathbb{R}$ is the only open set containing H and $H = U \cap V$ where U is open and V is β-$\omega^\star$-regular. Then $H \subset U$. So $U = \mathbb{R}$. Thus $H = \mathbb{R} \cap V = V$ where V is β-$\omega^\star$-regular which implies H is β-$\omega^\star$-regular. This is a contradiction since $H = \mathbb{Q}$ is not β-$\omega^\star$-regular by (4) of Example 4.14. This concludes that H is not a weak ω-$\mathcal{AB}$-set.

(2) $H = \mathbb{Q}^\star$ is open and hence a weak ω-$\mathcal{AB}$-set. But $H = \mathbb{Q}^\star$ is not an ω-t_α-set by (3) of Example 4.11.

THEOREM 4.4. *For a subset H of a space (X, τ), the following are equivalent:*

(1) *H is open;*

(2) *H is α-ω-open and a weak ω-$\mathcal{AB}$-set.*

PROOF. (1)⇒(2): Proof follows directly by Remark 2.1 and by (1) of Proposition 4.8.

(2)⇒(1): Given H is α-ω-open and a weak ω-$\mathcal{AB}$-set. Since H is a weak ω-$\mathcal{AB}$-set, $H = U \cap V$ where U is open and V is β-$\omega^\star$-regular. Then $H \subset U = int(U)$ and $H \subset V$. Also H is α-ω-open implies $H \subset int_\omega(cl(int_\omega(H))) \subset int_\omega(cl(int_\omega(V))) = int(V)$ for V is an ω-t_α-set being β-$\omega^\star$-regular. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open. □

REMARK 4.12. The notions of α-ω-openness and being a weak ω-$\mathcal{AB}$-set are independent.

EXAMPLE 4.19. In $\mathbb{R}$ with usual topology τ_u,

(1) $H = \mathbb{Q}^\star$ is α-ω-open. But H is not weak ω-$\mathcal{AB}$-set by (2) of Example 4.15.
(2) $H = (0, 1]$ is a weak ω-$\mathcal{AB}$-set by (1) of Example 4.15. But H is not α-ω-open.

5. Decompositions of continuity

DEFINITION 5.1. [**3**] A function $f : X \to Y$ is said to be pre-ω-continuous (resp. α-ω-continuous) if $f^{-1}(V)$ is pre-ω-open (resp. α-ω-open) in X for each open set V in Y.

DEFINITION 5.2. A function $f : X \to Y$ is said to be semi-ω-continuous (resp. b-ω-continuous, β-ω-continuous, $\omega^{\#}$-$\mathcal{B}$-continuous, $\omega^{\#}$-$\mathcal{B}_\alpha$-continuous, ω-$\mathcal{SB}$-continuous, ω-$\mathcal{SC}$-continuous, strongly ω-$\mathcal{B}$-continuous, ω-t_α-continuous, β-$\omega^\star$-regular continuous, weakly ω-$\mathcal{AB}$-continuous) if $f^{-1}(V)$ is semi-ω-open (resp. b-ω-open, β-ω-open, an $\omega^{\#}$-$\mathcal{B}$-set, an $\omega^{\#}$-$\mathcal{B}_\alpha$-set, an ω-$\mathcal{SB}$-set, an ω-$\mathcal{SC}$-set, a strong ω-$\mathcal{B}$-set, an ω-t_α-set, β-$\omega^\star$-regular, a weak ω-$\mathcal{AB}$-set) in X for each open set V in Y.

THEOREM 5.1. *For a function $f : X \to Y$, the following are equivalent:*
(1) *f is continuous.*
(2) *f is semi-ω-continuous and $\omega^{\#}$-$\mathcal{B}$-continuous.*

PROOF. This is an immediate consequence of Theorem 3.1. □

THEOREM 5.2. *For a function $f : X \to Y$, the following are equivalent:*
(1) *f is continuous.*
(2) *f is β-ω-continuous and $\omega^{\#}$-$\mathcal{B}_\alpha$-continuous.*

PROOF. This is an immediate consequence of Theorem 3.2. □

THEOREM 5.3. *For a function $f : X \to Y$, the following are equivalent:*
(1) *f is continuous.*
(2) *f is b-ω-continuous and strongly ω-$\mathcal{B}$-continuous.*

PROOF. This is an immediate consequence of Theorem 3.3. □

THEOREM 5.4. *For a function $f : X \to Y$, the following are equivalent:*
(1) *f is continuous.*
(2) *f is pre-ω-continuous and ω-$\mathcal{SB}$-continuous.*

PROOF. This is an immediate consequence of Theorem 4.1. □

THEOREM 5.5. *For a function $f : X \to Y$, the following are equivalent:*
(1) *f is continuous.*
(2) *f is semi-ω-continuous and ω-$\mathcal{SC}$ continuous.*

PROOF. This is an immediate consequence of Theorem 4.2. □

THEOREM 5.6. *For a function $f : X \to Y$, the following are equivalent.*
(1) *f is β-$\omega^\star$-regular continuous.*

(2) *f is ω-t_α-continuous and weakly ω-$\mathcal{AB}$-continuous.*

PROOF. This is an immediate consequence of Theorem 4.3. □

THEOREM 5.7. *For a function $f : X \to Y$, the following are equivalent:*
(1) *f is continuous.*
(2) *f is α-ω-continuous and weakly ω-$\mathcal{AB}$-continuous.*

PROOF. This is an immediate consequence of Theorem 4.4. □

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[1,3]DEPARTMENT OF MATHEMATICS, P. M. THEVAR COLLEGE, USILAMPATTI, MADURAI DT, TAMIL NADU, INDIA.
E-mail address: siingam@yahoo.com and rajasekarani@yahoo.com

[2]DEPARTMENT OF MATHEMATICS, SREE SOWDAMBIGA COLLEGE OF ENGINEERING,, ARUPPUKOTTAI, VIRUDHUNAGAR DT, TAMIL NADU, INDIA.
E-mail address: jothimp123@gmail.com

[4]DEPARTMENT OF MATHEMATICS, RAJAH SERFOJI GOVT. COLLEGE (AUTONOMOUS), THANJAVUR, TAMIL NADU, INDIA
E-mail address: satheesh_kanna@yahoo.co.in