

Common Coupled Fixed Points for Four Maps Using Property(E.A.) in Complex Valued b - Metric Spaces

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Abstract In this paper, we obtain a unique common coupled fixed point theorem for four self maps using property(E.A.) in complex valued b -metric spaces. Also we give an example to illustrate our main theorem.

Keywords: Complete metric spaces, property(E.A), w -compatible mappings

1 Introduction and Preliminaries

It is a well-known fact that the mathematical results regarding fixed points of contraction type mappings are very useful for determining the existence and uniqueness of solutions to various mathematical models.

Azam et al.[1] introduced the notion of a complex valued metric space which is a generalization of the classical metric space and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a rational contractive condition. Later several authors proved fixed and common fixed point theorems in complex valued metric spaces, for example, refer to [2,3,4,6,9,10,13,15,16,18,25]. Recently some authors, for example [5,12,14,19,21,22,23], obtained coupled and common coupled fixed point theorems for a pair of mappings in complex valued metric spaces.

In this paper, we prove a unique common coupled fixed point theorem for two pairs of mappings satisfying a contractive condition of rational type in the frame work of complex valued b -metric spaces using property(E.A.).

To begin, we recall some basic definitions, notations and results.

Throughout this paper \mathcal{R} , \mathcal{R}^+ , \mathcal{N} and \mathbb{C} denote the set of all real numbers, non-negative real numbers, positive integers and complex numbers respectively. First we refer to the following preliminaries.

Let $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows: $z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$. Thus $z_1 \preceq z_2$ if one of the following holds:

- (1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
- (2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
- (3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
- (4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Clearly $z_1 \preceq z_2 \Rightarrow |z_1| \leq |z_2|$.

Definition 1.1 ([1]) Let X be a non empty set. A function $d : X \times X \rightarrow \mathbb{C}$ is called a complex valued metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

- (i) $0 \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$;
- (iii) $d(x, y) \preceq d(x, z) + d(z, y)$.

The pair (X, d) is called a complex valued metric space.

Now, we briefly review the notation about complex valued b -metric spaces.

Definition 1.2 ([7]) Let X be a non empty set and $s \geq 1$. A function $d : X \times X \rightarrow \mathbb{C}$ is called a complex valued b - metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

- (i) $0 \lesssim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$;
- (iii) $d(x, y) \lesssim s [d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b - metric space.

Remark 1.3 Let (X, d) be a complex valued b -metric space with $s \geq 1$. Then

- (i) $|d(x, y)|$ or $|d(u, v)| < |1 + s d(x, y) + s d(u, v)|, \forall x, y, u, v \in X$.
- (ii) If $x \neq y$ then $|d(x, y)| > 0$.
- (iii) For $0 \leq k < 1$ and $z, w \in \mathbb{C}$, if $|z| \leq k|w|$ and $|w| \leq k|z|$ then $z = w = 0$.

Definition 1.4 ([7]) Let (X, d) be a complex valued b -metric space.

- (i) A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.
- (ii) A point $x \in X$ is called a limit point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) \cap (X - A) \neq \phi$.
- (iii) A subset $B \subseteq X$ is called open whenever each point of B is an interior point of B .
- (iv) A subset $B \subseteq X$ is called closed whenever each limit point of B is in B .
- (v) The family $F = \{B(x, r) : x \in X \text{ and } 0 \prec r\}$ is a sub basis for a topology on X . We denote this complex topology by τ_c . Indeed, the topology τ_c is Hausdorff.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \preceq c$ there is $n_0 \in \mathcal{N}$ such that for all $n > n_0, d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathcal{N}$ such that for all $n > n_0, d(x_n, x_{n+m}) \prec c$, where $m \in \mathcal{N}$, then $\{x_n\}$ is called a Cauchy sequence in (X, d) . If every Cauchy sequence is convergent in (X, d) then (X, d) is called a complete complex valued b -metric space. We require the following lemmas.

Lemma 1.5 ([7]) Let (X, d) be a complex valued b -metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 1.6 ([7]) Let (X, d) be a complex valued b -metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \rightarrow 0$ as $n, m \rightarrow \infty$.

One can easily prove the following lemma

Lemma 1.7 Let (X, d) be a complex valued b -metric space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X converging to x and y respectively. Then

- (i) $\frac{1}{s} |d(x, z)| \leq \lim_{n \rightarrow \infty} |d(x_n, z)| \leq s |d(x, z)|$ for all $z \in X$,
- (ii) $\frac{1}{s^2} |d(x, y)| \leq \lim_{n \rightarrow \infty} |d(x_n, y_n)| \leq s^2 |d(x, y)|$.

Recently Bhaskar and Lakshmikantham [20] introduced the concept of coupled fixed point and discussed some problems of the uniqueness of a coupled fixed point and applied their results to the problems of the existence and uniqueness of a solution for the periodic boundary value problems. Later Lakshmikantham and Ćirić [24] proved some coupled coincidence and coupled common fixed point results in partially ordered metric spaces.

Definition 1.8 Let X be a non empty set and $F : X \times X \rightarrow X$ and $S : X \rightarrow X$.

- (i) ([24]) An element $(x, y) \in X \times X$ is called a common coupled fixed point of mappings F and S if $x = Sx = F(x, y)$ and $y = Sy = F(y, x)$.
- (ii) ([8]) The pair (F, S) is called w -compatible if $S(F(x, y)) = F(Sx, Sy)$ and $S(F(y, x)) = F(Sy, Sx)$, whenever $Sx = F(x, y)$ and $Sy = F(y, x)$.

Verma and Pathak [17] defined property (E.A.), like [11] in complex valued metric spaces as follows.

Definition 1.9 ([17]) Let $A, S : X \rightarrow X$ be two self-maps of a complex-valued metric space (X, d) . The pair (A, S) is said to satisfy property (E.A.), if there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t$ for some $t \in X$.

In this paper, we define property (E.A.) for two mappings $F : X \times X \rightarrow X$ and $S : X \rightarrow X$ as follows in complex valued metric spaces.

Definition 1.10 Let X be a nonempty set and $F : X \times X \rightarrow X$ and $S : X \rightarrow X$ be two mappings. Then the pair (F, S) is said to satisfy property (E.A.) if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that for some $t, t' \in X$, we have $\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} Sx_n = t$ and $\lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} Sy_n = t'$.

2 Main Result

Theorem 2.1 Let (X, d) be a complex valued b -metric space with $s \geq 1$. Let $F, G : X \times X \rightarrow X$ and $S, T : X \rightarrow X$ be mappings satisfying the following :

(2.1.1)

$$\begin{aligned}
 & s^3 d(F(x, y), G(u, v)) \\
 & \lesssim a_1 d(Sx, Tu) + a_2 d(Sy, Tv) + a_3 d(Sx, F(x, y)) \\
 & \quad + a_4 d(Sy, F(y, x)) + a_5 d(Tu, G(u, v)) + a_6 d(Tv, G(v, u)) \\
 & \quad + a_7 d(Sx, G(u, v)) + a_8 d(Sy, G(v, u)) + a_9 d(Tu, F(x, y)) + a_{10} d(Tv, F(y, x)) \\
 & \quad + a_{11} \frac{d(Sx, F(x, y)) d(Tu, G(u, v))}{1+s d(Sx, Tu)+s d(Sy, Tv)} + a_{12} \frac{d(Tu, F(x, y)) d(Sx, G(u, v))}{1+s d(Sx, Tu)+s d(Sy, Tv)}
 \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i, i = 1, 2, \dots, 12$ are non-negative real numbers such that $\sum_{i=1}^{12} a_i < \frac{1}{s}$.

(2.1.2) The pairs (F, S) and (G, T) are w -compatible.

(2.1.3)(a) The pair (F, S) satisfies property (E.A.), $F(X \times X) \subseteq T(X)$ and $S(X)$ is a closed sub space of X .

(or)

(2.1.3)(b) The pair (G, T) satisfies property (E.A.), $G(X \times X) \subseteq S(X)$ and $T(X)$ is a closed sub space of X .

Then F, G, S and T have a unique common coupled fixed point.

Proof. Suppose (2.1.3)(a) holds.

Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} Sx_n = t$ and

$\lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} Sy_n = t'$ for some t and t' in X .

Since $F(X \times X) \subseteq T(X)$, there exist sequences $\{z_n\}$ and $\{w_n\}$ in X such that $F(x_n, y_n) = Tz_n \rightarrow t$ and $F(y_n, x_n) = Tw_n \rightarrow t'$ as $n \rightarrow \infty$.

Consider

$$\begin{aligned}
 & s |d(t, G(z_n, w_n))| \\
 & \leq s^2 |d(t, F(x_n, y_n))| + s^2 |d(F(x_n, y_n), G(z_n, w_n))| \\
 & \leq s^2 |d(t, F(x_n, y_n))| + s^3 |d(F(x_n, y_n), G(z_n, w_n))|, \text{ since } s \geq 1 \\
 & \leq s^2 |d(t, F(x_n, y_n))| + a_1 |d(Sx_n, Tz_n)| + a_2 |d(Sy_n, Tw_n)| \\
 & \quad + a_3 |d(Sx_n, F(x_n, y_n))| + a_4 |d(Sy_n, F(y_n, x_n))| + a_5 |d(Tz_n, G(z_n, w_n))| \\
 & \quad + a_6 |d(Tw_n, G(w_n, z_n))| + a_7 |d(Sx_n, G(z_n, w_n))| + a_8 |d(Sy_n, G(w_n, z_n))| \\
 & \quad + a_9 |d(Tz_n, F(x_n, y_n))| + a_{10} |d(Tw_n, F(y_n, x_n))| \\
 & \quad + a_{11} \frac{|d(Sx_n, F(x_n, y_n))| |d(Tz_n, G(z_n, w_n))|}{|1+sd(Sx_n, Tz_n)+sd(Sy_n, Tw_n)|} \\
 & \quad + a_{12} \frac{|d(Sx_n, G(z_n, w_n))| |d(Tz_n, F(x_n, y_n))|}{|1+sd(Sx_n, Tz_n)+sd(Sy_n, Tw_n)|}.
 \end{aligned}$$

Letting $n \rightarrow \infty$ and using Lemma 1.7, we get

$$\begin{aligned} s \lim_{n \rightarrow \infty} |d(t, G(z_n, w_n))| &\leq s^2(0) + a_1(0) + a_2(0) + a_3(0) + a_4(0) + a_5 s \lim_{n \rightarrow \infty} |d(t, G(z_n, w_n))| \\ &\quad + a_6 s \lim_{n \rightarrow \infty} |d(t', G(w_n, z_n))| + a_7 s \lim_{n \rightarrow \infty} |d(t, G(z_n, w_n))| \\ &\quad + a_8 s \lim_{n \rightarrow \infty} |d(t', G(w_n, z_n))| + a_9(0) + a_{10}(0) + a_{11}(0) + a_{12}(0) \\ &= (a_5 + a_7) s \lim_{n \rightarrow \infty} |d(t, G(z_n, w_n))| + (a_6 + a_8) s \lim_{n \rightarrow \infty} |d(t', G(w_n, z_n))| \end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} |d(t, G(z_n, w_n))| \leq \frac{a_6 + a_8}{1 - a_5 - a_7} \lim_{n \rightarrow \infty} |d(t', G(w_n, z_n))|.$$

Similarly, we can show that

$$\lim_{n \rightarrow \infty} |d(t', G(w_n, z_n))| \leq \frac{a_6 + a_8}{1 - a_5 - a_7} \lim_{n \rightarrow \infty} |d(t, G(z_n, w_n))|.$$

Hence $\lim_{n \rightarrow \infty} |d(t, G(z_n, w_n))| = 0 = \lim_{n \rightarrow \infty} |d(t', G(w_n, z_n))|$.

Thus $G(z_n, w_n) \rightarrow t$ and $G(w_n, z_n) \rightarrow t'$, by Lemma 1.5.

Since $S(X)$ is a closed sub space of X , there exist $u, v \in X$ such that

$$\lim_{n \rightarrow \infty} Sx_n = t = Su \text{ and } \lim_{n \rightarrow \infty} Sy_n = t' = Sv.$$

Using(2.1.1) and Lemma 1.7, we have

$$\begin{aligned} s |d(F(u, v), Su)| &= s^2 \frac{1}{s} |d(F(u, v), Su)| \\ &\leq s^3 \lim_{n \rightarrow \infty} |d(F(u, v), G(z_n, w_n))|, \text{ since } s \geq 1 \\ &\leq \lim_{n \rightarrow \infty} \left\{ \begin{aligned} &a_1 |d(Su, Tz_n)| + a_2 |d(Sv, Tw_n)| + a_3 |d(Su, F(u, v))| + a_4 |d(Sv, F(v, u))| \\ &+ a_5 |d(Tz_n, G(z_n, w_n))| + a_6 |d(Tw_n, G(w_n, z_n))| + a_7 |d(Su, G(z_n, w_n))| \\ &+ a_8 |d(Sv, G(w_n, z_n))| + a_9 |d(Tz_n, F(u, v))| + a_{10} |d(Tw_n, F(v, u))| \\ &+ a_{11} \frac{|d(Su, F(u, v))| |d(Tz_n, G(z_n, w_n))|}{1 + sd(Su, Tz_n) + sd(Sv, Tw_n)} + a_{12} \frac{|d(Su, G(z_n, w_n))| |d(Tz_n, F(u, v))|}{1 + sd(Su, Tz_n) + sd(Sv, Tw_n)} \end{aligned} \right\} \\ &\leq a_3 |d(Su, F(u, v))| + a_4 |d(Sv, F(v, u))| + a_9 s |d(Su, F(u, v))| + a_{10} s |d(Sv, F(v, u))| \\ &\leq s(a_3 + a_9) |d(Su, F(u, v))| + s(a_4 + a_{10}) |d(Sv, F(v, u))|, \text{ since } s \geq 1 \end{aligned}$$

Thus $|d(F(u, v), Su)| \leq \frac{(a_4 + a_{10})}{1 - a_3 - a_9} |d(Sv, F(v, u))|$.

Similarly, we can show that $|d(F(v, u), Sv)| \leq \frac{(a_4 + a_{10})}{1 - a_3 - a_9} |d(Su, F(u, v))|$. Hence $F(u, v) = Su = t$ and $F(v, u) = Sv = t' \dots (1)$

Since (F, S) is w -compatible, we have $F(t, t') = St$ and $F(t', t) = St' \dots (2)$

Since $F(X \times X) \subseteq T(X)$, there exist $\alpha, \beta \in X$ such that $t = F(u, v) = T\alpha$ and $t' = F(v, u) = T\beta$.

$$\begin{aligned} d(t, G(\alpha, \beta)) &\leq s^3 d(F(u, v), G(\alpha, \beta)) \\ &\leq a_1 d(Su, T\alpha) + a_2 d(Sv, T\beta) + a_3 d(Su, F(u, v)) + a_4 d(Sv, F(v, u)) \\ &\quad + a_5 d(T\alpha, G(\alpha, \beta)) + a_6 d(T\beta, G(\beta, \alpha)) + a_7 d(Su, G(\alpha, \beta)) \\ &\quad + a_8 d(Sv, G(\beta, \alpha)) + a_9 d(T\alpha, F(u, v)) + a_{10} d(T\beta, F(v, u)) \\ &\quad + a_{11} \frac{d(Su, F(u, v)) d(T\alpha, G(\alpha, \beta))}{1 + sd(Su, T\alpha) + sd(Sv, T\beta)} + a_{12} \frac{d(T\alpha, F(u, v)) d(Sv, G(\beta, \alpha))}{1 + sd(Su, T\alpha) + sd(Sv, T\beta)} \\ &= a_5 d(t, G(\alpha, \beta)) + a_6 d(t', G(\beta, \alpha)) + a_7 d(t, G(\alpha, \beta)) + a_8 d(t', G(\beta, \alpha)) \\ &= (a_5 + a_7) d(t, G(\alpha, \beta)) + (a_6 + a_8) d(t', G(\beta, \alpha)). \end{aligned}$$

Thus $|d(t, G(\alpha, \beta))| \leq \frac{a_6 + a_8}{1 - a_5 - a_7} |d(t', G(\beta, \alpha))|$.

Similarly, we can show that $|d(t', G(\beta, \alpha))| \leq \frac{a_6 + a_8}{1 - a_5 - a_7} |d(t, G(\alpha, \beta))|$.

Hence $G(\alpha, \beta) = t = T\alpha$ and $G(\beta, \alpha) = t' = T\beta$.

Since (G, T) is w -compatible, we have $G(t, t') = Tt$ and $G(t', t) = Tt' \dots (3)$

Consider

$$\begin{aligned} &1 + d(St, t) + d(St', t') \\ &\quad \lesssim 1 + sd(St, Tz_n) + sd(Tz_n, t) + sd(St', Tw_n) + sd(Tw_n, t') \\ &|1 + d(St, t) + d(St', t')| \\ &\leq |1 + sd(St, Tz_n) + sd(St', Tw_n)| + s |d(Tz_n, t)| + s |d(Tw_n, t')|. \end{aligned}$$

Letting $n \rightarrow \infty$ and using Lemma 1.5, we get

$$|1 + d(St, t) + d(St', t')| \leq \lim_{n \rightarrow \infty} |1 + sd(St, Tz_n) + sd(St', Tw_n)| \dots\dots(4)$$

From(2.1.1) and using Lemma 1.7, we get

$$\begin{aligned} s^2 |d(St, t)| &= s^3 \frac{1}{s} |d(F(t, t'), t)| \leq s^3 \lim_{n \rightarrow \infty} |d(F(t, t'), G(z_n, w_n))| \\ &\leq \lim_{n \rightarrow \infty} \left[\begin{aligned} &a_1 |d(St, Tz_n)| + a_2 |d(St', Tw_n)| + a_3 |d(St, F(t, t'))| + a_4 |d(St', F(t', t))| \\ &+ a_5 |d(Tz_n, G(z_n, w_n))| + a_6 |d(Tw_n, G(w_n, z_n))| + a_7 |d(St, G(z_n, w_n))| \\ &+ a_8 |d(St', G(w_n, z_n))| + a_9 |d(Tz_n, F(t, t'))| + a_{10} |d(Tw_n, F(t', t))| + \\ &a_{11} \frac{|d(St, F(t, t'))| |d(Tz_n, G(z_n, w_n))|}{|1 + sd(St, Tz_n) + sd(St', Tw_n)|} + a_{12} \frac{|d(St, G(z_n, w_n))| |d(Tz_n, F(t, t'))|}{|1 + sd(St, Tz_n) + sd(St', Tw_n)|} \end{aligned} \right] \\ &\leq a_1 s |d(St, t)| + a_2 s |d(St', t')| + a_7 s |d(St, t)| + a_8 s |d(St', t')| + a_9 s |d(St, t)| \\ &\quad + a_{10} s |d(St', t')| + a_{12} \frac{s |d(St, t)| |s |d(St, t)|}{|1 + d(St, t) + d(St', t')|} \\ &\leq (a_1 + a_7 + a_9 + a_{12}) s^2 |d(St, t)| + (a_2 + a_8 + a_{10}) s^2 |d(St', t')|. \end{aligned}$$

Thus $|d(St, t)| \leq \frac{a_2 + a_8 + a_{10}}{1 - a_1 - a_7 - a_9 - a_{12}} |d(St', t')|$.

Similarly, we can show that $|d(St', t')| \leq \frac{a_2 + a_8 + a_{10}}{1 - a_1 - a_7 - a_9 - a_{12}} |d(St, t)|$.

Hence $St = t$ and $St' = t'$. Thus from (2),

$$F(t, t') = St = t \text{ and } F(t', t) = St' = t' \dots\dots(5)$$

Similarly, consider

$$\begin{aligned} s^2 |d(t, Tt)| &= s^3 \frac{1}{s} |d(t, G(t, t'))| \leq s^3 \lim_{n \rightarrow \infty} |d(F(x_n, y_n), G(t, t'))| \\ &\leq \lim_{n \rightarrow \infty} \left[\begin{aligned} &a_1 |d(Sx_n, Tt)| + a_2 |d(Sy_n, Tt')| + a_3 |d(Sx_n, F(x_n, y_n))| + a_4 |d(Sy_n, F(y_n, x_n))| \\ &+ a_5 |d(Tt, G(t, t'))| + a_6 |d(Tt', G(t', t))| + a_7 |d(Sx_n, G(t, t'))| \\ &+ a_8 |d(Sy_n, G(t', t))| + a_9 |d(Tt, F(x_n, y_n))| + a_{10} |d(Tt', F(y_n, x_n))| + \\ &a_{11} \frac{|d(Sx_n, F(x_n, y_n))| |d(Tt, G(t, t'))|}{|1 + sd(Sx_n, Tt) + sd(Sy_n, Tt')|} + a_{12} \frac{|d(Sx_n, G(t, t'))| |d(Tt, F(x_n, y_n))|}{|1 + sd(Sx_n, Tt) + sd(Sy_n, Tt')|} \end{aligned} \right] \\ &\leq a_1 s |d(t, Tt)| + a_2 s |d(t', Tt')| + a_7 s |d(t, Tt)| + a_8 s |d(t', Tt')| + a_9 s |d(t, Tt)| \\ &\quad + a_{10} s |d(t', Tt')| + a_{12} \frac{s |d(t, Tt)| |s |d(t, Tt)|}{|1 + d(t, Tt) + d(t', Tt')|} \\ &\leq (a_1 + a_7 + a_9 + a_{12}) s^2 |d(t, Tt)| + (a_2 + a_8 + a_{10}) s^2 |d(t', Tt')|. \end{aligned}$$

Thus $|d(t, Tt)| \leq \frac{a_2 + a_8 + a_{10}}{1 - a_1 - a_7 - a_9 - a_{12}} |d(t', Tt')|$.

Similarly, we can show that $|d(t', Tt')| \leq \frac{a_2 + a_8 + a_{10}}{1 - a_1 - a_7 - a_9 - a_{12}} |d(t, Tt)|$.

Hence $Tt = t$ and $Tt' = t'$. Thus from (3),

$$G(t, t') = Tt = t \text{ and } G(t', t) = Tt' = t' \dots\dots(6)$$

From (5) and (6), it follows that (t, t') is a common coupled fixed point of F, G, S and T . Uniqueness of common coupled fixed point follows easily from (2.1.1). Similarly, we can prove the theorem whenever (2.1.3)(b) holds.

Now we give an example to illustrate Theorem 2.1

Example 2.2 Let $X = [0, 1]$ and $d(x, y) = i |x - y|^2, \forall x, y \in X$. Then $s = 2$.

Define $F, G : X \times X \rightarrow X$ by $F(x, y) = \frac{x^2 + y^2}{12}$ and $G(x, y) = \frac{x + y}{24}$ and $S, T : X \rightarrow X$ by $Sx = x^2$ and $Tx = \frac{x}{2}$.

Then $F(X \times X) = [0, \frac{1}{6}] \subseteq [0, \frac{1}{2}] = T(X)$ and $T(X)$ is a closed subspace of X .

Let $x_n = \frac{1}{n}, y_n = \frac{1}{\sqrt{n}} \in X$. Then

$\lim_{n \rightarrow \infty} F(x_n, y_n) = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} F(y_n, x_n) = \lim_{n \rightarrow \infty} Sy_n = 0$. Thus (2.1.3)(a) holds. Clearly the pairs (F, S) and (G, T) are w -compatible. Now consider

$$\begin{aligned} &s^3 d(F(x, y), G(u, v)) \\ &= 8i \left| \frac{x^2 + y^2}{12} - \frac{u + v}{24} \right|^2 \\ &= \frac{8}{576} i |2x^2 - u + 2y^2 - v|^2 \\ &\leq \frac{1}{72} 2i [|2x^2 - u|^2 + |2y^2 - v|^2] \\ &= \frac{i}{9} \left[\left| x^2 - \frac{u}{2} \right|^2 + \left| y^2 - \frac{v}{2} \right|^2 \right] \\ &= \frac{1}{9} [d(Sx, Tu) + d(Sy, Tv)] \end{aligned}$$

Thus (2.1.1) is satisfied with $a_1 = a_2 = \frac{1}{9}$ and $a_i = 0$ for $i = 3, 4, \dots, 12$. Thus all the conditions of Theorem 2.1 are satisfied and $(0, 0)$ is the unique common coupled fixed point of F, G, S and T .

If $s = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.3 . Let (X, d) be a complex valued metric space. Let $F, G : X \times X \rightarrow X$ and $S, T : X \rightarrow X$ be mappings satisfying the following :

(2.3.1)

$$\begin{aligned} d(F(x, y), G(u, v)) & \lesssim a_1 d(Sx, Tu) + a_2 d(Sy, Tv) + a_3 d(Sx, F(x, y)) \\ & + a_4 d(Sy, F(y, x)) + a_5 d(Tu, G(u, v)) + a_6 d(Tv, G(v, u)) \\ & + a_7 d(Sx, G(u, v)) + a_8 d(Sy, G(v, u)) + a_9 d(Tu, F(x, y)) + a_{10} d(Tv, F(y, x)) \\ & + a_{11} \frac{d(Sx, F(x, y)) d(Tu, G(u, v))}{1+d(Sx, Tu)+d(Sy, Tv)} + a_{12} \frac{d(Tu, F(x, y)) d(Sx, G(u, v))}{1+d(Sx, Tu)+d(Sy, Tv)} \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i, i = 1, 2, \dots, 12$ are non-negative real numbers such that $\sum_{i=1}^{12} a_i < 1$.

(2.3.2) The pairs (F, S) and (G, T) are w -compatible.

(2.3.3)(a) The pair (F, S) satisfies property(E.A.), $F(X \times X) \subseteq T(X)$ and $S(X)$ is a closed sub space of X .

(or)

(2.3.3)(b) The pair (G, T) satisfies property(E.A.), $G(X \times X) \subseteq S(X)$ and $T(X)$ is a closed sub space of X .

Then F, G, S and T have a unique common coupled fixed point.

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