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NEW PERSPECTIVES IN THE METRIC THEORY OF CONTINUED FRACTION EXPANSION RELATED TO FIBONACCI TYPE SEQUENCES

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***Abstract**: A survey of the metric theory of the continued fraction expansions related to random Fibonacci Type sequences discussed by Sebe and Lascu is given. The limit properties of these expansions have been studied. A Wirsing-type approach to the Perron-Frobenius operator of the generalized Gauss map under its invariant measure allows us to get close to the optimal convergence rate. Actually, we obtain upper and lower bounds of the convergence rate which provide a near-optimal solution to the Gauss-Kuzmin-Lévy problem for these expansions.*

***Keywords:** continued fractions, random Fibonacci-type sequences, Perron-Frobenius operator, random system with complete connections, Gauss-Kuzmin-Lévy problem*

INTRODUCTION

In this paper we consider a non-regular continued fraction expansions introduced by Chan [1].
In fact, Chan considered some continued fraction expansions related to random Fibonacci-type sequences which were studied in detail by Sebe and Lascu in [6, 4, 5].

PREREQUISITES. Fix an integer $m \geq 2$. Define on $I := [0,1]$ the transformation τ_m by $\tau_m : I \to I$,

$$\tau_m(x) := \begin{cases} \frac{1}{m-1}\left(\frac{1}{m^i x} - 1\right), & x \in I_i, \\ 0, & x = 0 \end{cases} \tag{1}$$

where $I_i := \left\{x \in I : m^{-(i+1)} < x \leq m^{-i}\right\}$ for $i \in \mathbb{N} := \{0,1,2,\ldots\}$. For any $x \in (0,1)$ put $a_n = a_n(x) = a_1\left(\tau_m^{n-1}(x)\right)$, $n \in \mathbb{N}_+ := \{1,2,\ldots\}$, with $\tau_m^0(x) = x$ and $a_1 = a_1(x) = \begin{cases} \lfloor \log x^{-1} / \log m \rfloor, & x \neq 0, \\ \infty, & x = 0. \end{cases}$

Then every $x \in [0,1)$ has an infinite expansion

$$x = \cfrac{m^{-a_1}}{1 + \cfrac{(m-1)m^{-a_2}}{1 + \cfrac{(m-1)m^{-a_3}}{1 + \ddots}}} := [a_1, a_2, \ldots]_m, \tag{2}$$

where a_n's are non-negative integers. The numbers $p_n(x)/q_n(x) = [a_1, a_2, \ldots]_m$ are the n-th order convergent of $x \in [0,1)$. Then $p_n(x)/q_n(x) \to x, n \to \infty$. Here p_n's and q_n's satisfy for $n \in \mathbb{N}_+$ the following:

$p_n(x) = m^{a_n} p_{n-1}(x) + (m-1)m^{a_{n-1}} p_{n-2}(x), n \geq 2$,
$q_n(x) = m^{a_n} q_{n-1}(x) + (m-1)m^{a_{n-1}} q_{n-2}(x), n \geq 1$,

with $p_0(x) = 0, q_0(x) = 1, p_1(x) = 1, q_{-1}(x) = 0$ and $a_0 = 0$. In [1] it was shown that the invariant probability measure of the transformation τ_m is

$$\gamma_m(A) := k_m \int_A \frac{dx}{((m-1)x+1)((m-1)x+m)}, \tag{3}$$

$A \in \mathcal{B}_I$, where $\mathcal{B}_I$ is the σ- algebra of the Borel subsets of I. Hence, $\gamma_m(A) = \gamma_m\left(\tau_m^{-1}(A)\right), A \in \mathcal{B}_I$, the sequence $(a_n)_{n\in\mathbb{N}_+}$ is strictly stationary on $(I, \mathcal{B}_I, \gamma_m)$.

METRIC PROPERTIES. Roughly speaking, the metrical theory of continued fraction expansions is about the sequence $(a_n)_{n\in\mathbb{N}_+}$ and related sequences. For any $n \in \mathbb{N}_+$ and $i^{(n)} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ we will say that $I_m\left(i^{(n)}\right) = \{\omega \in \Omega : a_k(\omega) = i_k, 1 \leq k \leq n\}$ is the fundamental interval of rank n and make the convention that $I_m\left(i^{(0)}\right) = \Omega$, where Ω is the set of irrationals in I. We will write $I_m(a_1, \ldots, a_n) = I_m\left(a^{(n)}\right)$, $n \in \mathbb{N}_+$. If $n \geq 2$ and $i_n \in \mathbb{N}$, then we have $I_m(a_1, \ldots, a_n) = I_m\left(i^{(n)}\right)$. For any $n \in \mathbb{N}_+$ we have

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$$\lambda\left(\tau_m^n < x \middle| a_1,\ldots,a_n\right) = \frac{(s_n+m)x}{s_n+(m-1)x+1},\ x\in I \quad (4)$$

where $s_n = m^{-a_n}\dfrac{q_n}{q_{n-1}} - 1$, $s_1 = 0$. Equation (4) is the Brodén-Borel-Lévy formula for this type of expansions. It allows us to determine the probability distribution of $(a_n)_{n\in\mathbb{N}_+}$ under the Lebesgue measure λ. Clearly,

$\lambda(a_1 = i) = (m-1)m^{-(i+1)}, i\in\mathbb{N}$ and $n\in\mathbb{N}_+$ and $\lambda\left(a_{n+1} = i \middle| a_1,\ldots,a_n\right) = P_{m,i}(s_n)$, where

$$P_{m,i}(x) = \frac{(m-1)m^{-(i+1)}(x+1)(x+m)}{\left(x+(m-1)m^{-i}+1\right)\left(x+(m-1)m^{-(i+1)}+1\right)}. (5).$$

We have already noticed that the sequence $(a_n)_{n\in\mathbb{N}_+}$ is strictly stationary on (I,B_I,γ_m). As such, a doubly infinite version of it, say $(\bar{a}_l)_{l\in\mathbb{Z}}$ should exist on a richer probability space. Indeed, such a version can be effectively constructed on $\left(I^2,B_I^2,\bar{\gamma}_m\right)$ where $\bar{\gamma}_m$, the extended measure, is defined by

$$\bar{\gamma}_m(B) = k_m \iint_B \frac{dxdy}{\left((m-1)(x+y)+1\right)^2}, B\in B_I^2. \quad (6)$$

For $n\in\mathbb{N}_+$ and $(\omega,\theta)\in\Omega^2$, put $\bar{a}_n(\omega,\theta) = a_n(\omega)$, $\bar{a}_0(\omega,\theta) = a_1(\theta), \bar{a}_{-n}(\omega,\theta) = a_{n+1}(\theta)$. Then for any $l\in\mathbb{Z}, k\in\mathbb{N}$ and $n\in\mathbb{N}_+$ the probability distribution of the random vector $\left(\bar{a}_l,\ldots,\bar{a}_{l+k}\right)$ under $\bar{\gamma}_m$ is identical with that of the random vector $(a_n,\ldots,a_{n+k})$ under γ_m. In other words, the doubly infinite sequence $(\bar{a}_l)_{l\in\mathbb{Z}}$ is strictly stationary under $\bar{\gamma}_m$. The definition of $(\bar{a}_l)_{l\in\mathbb{Z}}$ is associated with the natural extension $\bar{\tau}_m$ of τ_m which is a transformation of $[0,1)\times I$ defined by

$$\bar{\tau}_m(x,y) = \left(\tau_m(x), \frac{m^{-a_1(x)}}{(m-1)y+1}\right),\ (x,y)\in[0,1)\times I.$$

This is a one-to-one transformation of Ω^2 with the inverse

$$\bar{\tau}_m^{-1}(\omega,\theta) = \left(\frac{m^{-a_1(\theta)}}{(m-1)\omega+1}, \tau_m(\theta)\right), (\omega,\theta)\in\Omega^2.$$

The extended measure $\bar{\gamma}_m$ is $\bar{\tau}_m$-invariant, that is, $\bar{\gamma}_m = \bar{\gamma}_m \bar{\tau}_m^{-1}$ and $\bar{a}_{l+1}(\omega,\theta) = \bar{a}_1\left(\bar{\tau}_m^{-l}(\omega,\theta)\right)$ with $\bar{a}_1(\omega,\theta) = a_1(\omega)$, $(\omega,\theta)\in\Omega^2$. The dependence structure of $(\bar{a}_l)_{l\in\mathbb{Z}}$ follows from the fact that

$\bar{\gamma}_m\left([0,x]\times I \middle| \bar{a}_0,\bar{a}_{-1},\ldots\right) = \dfrac{\left((m-1)a+m\right)x}{(m-1)(x+a)+1}$ $\bar{\gamma}_m$-a.s., for any $x\in I$, where $a = \left[\bar{a}_0,\bar{a}_{-1},\ldots\right]_m$. Hence

$\bar{\gamma}_m\left(\bar{a}_1 = i \middle| \bar{a}_0,\bar{a}_{-1},\ldots\right) = P_{m,i}\left((m-1)a\right)$ $\bar{\gamma}_m$-a.s., for any $i\in\mathbb{N}$. We thus see that $(\bar{a}_l)_{l\in\mathbb{Z}}$ is an infinite order chain in the theory of dependence with complete connections (see [2], section 5.5).

GAUSS-KUZMIN-TYPE THEOREM

It is only recently [6, 4, 5] that the limits and ergodic properties of these expansions have been studied. It should be stressed that the ergodic theorem does not yield rates of convergence for mixing properties; for this a Gauss-Kuzmin theorem is needed.

Limits properties. Let us consider the random system with complete connections RSCC ([2])

$$\left\{(I,B_I),\left(\mathbb{N}_+,P_I(\mathbb{N}_+)\right),u,P\right\}, \quad (7)$$

where

$u: I\times\mathbb{N}\to I$, $u(x,i) = u_{m,i}(x) = \dfrac{m^{-i}}{(m-1)x+1}$ $x\in I$

and P is the transition probability function from (I,B_I) to $\left(\mathbb{N}_+,P_I(\mathbb{N}_+)\right)$ given in (5). Here $P_I(\mathbb{N}_+)$ denotes the power set of $\mathbb{N}_+$. For any $a\geq 0$ put $s_{0,a} = a$ and $s_{n,a} = \dfrac{(m-1)m^{-a_n}}{1+s_{na1}^a}, n\in\mathbb{N}_+$, and consider the family of (conditional) probability measures $(\gamma_{m,a})_a$ on B_I defined by their distribution functions

$$\gamma_{m,a}([0,x]) = \frac{\left((m-1)a+m\right)x}{(m-1)(x+a)+1}, \quad x\in I, \quad a\geq 0.$$

Then $(s_{n,a})_{n\in\mathbb{N}_+}$ is a $I\cup\{a\}$-valued Markov chain on $(I,B_I,\gamma_{m,a})$ which starts from $s_{0,a} = a\geq 0$ and has the following transition mechanism: from state $s\in I\cup\{a\}$ the possible transitions are to any state $m^{-i}/\left((m-1)s+1\right)$ with the corresponding transition probability $P_{m,i}\left((m-1)s\right)$, $i\in\mathbb{N}$. Thus the transition operator (Perron-Frobenius operator) U_m of all Markov chains $(s_{n,a})_{n\in\mathbb{N}}$ for any bounded complex-valued measurable function f on I, is given by

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$$U_m f(x) = \sum_{i \in \mathbb{N}_+} P_{m,i}\left((m-1)x\right) f\left(u_{m,i}(x)\right) \qquad (8)$$

$f \in L^1_{\gamma_m}$, where $L^1_{\gamma_m} := \left\{ f : I \to \mathbb{R} \,\middle|\, \int_0^1 |f| \, d\gamma_m < \infty \right\}$.

It was investigated in [6, 4, 5] the Perron-Frobenius operator of the continued fraction transformation τ_m under different probability measures on $\mathcal{B}_I$. The asymptotic behavior of this operator is given by

$$\mu\left(\tau_m^{-n}(A)\right) = \int_A \frac{U_m^n f(x)}{\left((m-1)x+1\right)\left((m-1)x+m\right)} dx$$

for any $n \in \mathbb{N}$ and $A \in \mathcal{B}_I$, where $f(x) = \left((m-1)x+1\right)\left((m-1)x+m\right)h(x)$, $x \in I$. In the sequel the domain of U will be successively restricted to various Banach spaces. Recall that the variation $\operatorname{var}_A f$ of f on a subset A of I is defined as $\sup \sum_{i=1}^{k-1} \left| f(t_i) - f(t_{i-1}) \right|$ the supremum being taken over all $t_1 < \ldots < t_k \in A$ and $k \geq 2$. If $\operatorname{var} f = \operatorname{var}_I f < \infty$ then f is called a function of bounded variation. A variation $v(f)$ for $L^\infty(I, \mathcal{B}_I, \lambda)$, the collection of all classes of λ-essentially bounded measurable complex-valued λ-indistinguishable function on I is defined as $v(f) = \inf \operatorname{var} f$, the infimum being taken over all versions of . The set $BEV(I)$ is a Banach space under the norm $\|f\|_v := v(f) + \|f\|_1$, where $\|\cdot\|_1$ is the usual L^1_λ norm $\|f\|_1 = \int_I |f| \, d\lambda$. For proofs and more details see [4, 5].

Whatever $a \geq 0$ the Markov chain $\left(s_{n,a}\right)_{n \in \mathbb{N}}$ associated with the RSCC (7) has the transition operator U, with the transition probability function $Q_m(x, A) = \sum_{i \in W_m(x,A)} P_{m,i}(x)$, $x \in I$, $A \in \mathcal{B}_I$, where $W_m(x, A) = \left\{ i \in \mathbb{N} \,\middle|\, u_{m,i}(x) \in A \right\}$. Then $Q^n(\cdot,\cdot)$ will denote the n-step transition probability function of the same Markov chain. It was proved in [4] that the RSCC (7) is uniformly ergodic and its transition operator is regular with respect to the Banach space of Lipschitz functions. Now for a probability measure μ on $\mathcal{B}_I$ we may determine the limit of the sequence $\left(\mu\left(\tau_m^n < x\right)\right)_{n \in \mathbb{N}_+}$ as $n \to \infty$ and obtain the rate of this convergence, i.e., $\lim_{n \to \infty} \mu\left(\tau_m^n < x\right) = \frac{k_m}{(m-1)^2} \log \frac{m\left((m-1)x+1\right)}{(m-1)x+m}$, $x \in I$, where $k_m = \frac{(m-1)^2}{\log\left(m^2/(2m-1)\right)}$.

A WIRSING TYPE APPROACH. Using a Wirsing-type approach [7], in [6] it was obtained a better estimate of the convergence rate involved. The strategy was to restrict the domain of the Perron-Frobenius operator of τ_m under its invariant measure γ_m to the Banach space of functions which have a continuous derivative on I. Define a linear operator $V_m : C(I) \to C(I)$ by $V_m g = -\left(U_m f\right)'$, $g \in C(I)$, where $f' = g$. Since U_m is a Markov operator, V_m is well defined. It is easy to check that $\left(U_m^n f\right)' = (-1)^n V_m^n f'$, $n \in \mathbb{N}_+$, $f \in C^1(I)$. In [6] Sebe proved that there are positive constants $v_m < w_m < 1$ and a real-valued function $\varphi_m \in C(I)$ defined by

$$\varphi_m(x) = (m-1)e_m^2 \cdot \frac{(m^2-1)((m-1)(a+1)-e_m)x^2 + 2m((m-1)(a+1)+(m-2)e_m)x + m(m+1)e_m}{\left(((m-1)(a+1)-e_m x)+e_m\right)^2 \left(((m-1)(e_m+a+1)-me_m x)+me_m\right)^2}$$

$x \in I$, where the coefficient e_m is chosen such that the equation

$$E_m(a) = (m+1)(m-1)^3(a+1)^4 - \\ - m\left(m^2+2m-1\right)e_m^3(a+1) - m(2m-1)e_m^4 = 0, x \in I$$

has a unique solution $a_m \in I$. For this unique acceptable $a_m \in I$, we have $v_m \varphi_m \leq V_m \varphi_m \leq w_m \varphi_m$ $m \in \mathbb{N}$, $m \geq 2$. Next, putting $\alpha_m = \min_{x \in I} \frac{\varphi_m(x)}{\left(f_m\right)'(x)}$ and $\beta_m = \max_{x \in I} \frac{\varphi_m(x)}{\left(f_m\right)'(x)}$ for any $f_m \in C^1(I)$ such that $\left(f_m\right)' > 0$, we get

$$\frac{\alpha_m}{\beta_m} v_m^n \left(f_m\right)' \leq V_m^n \left(f_m\right)' \leq \frac{\beta_m}{\alpha_m} w_m^n \left(f_m\right)', \; n \in \mathbb{N}_+.$$

In Theorem 5.3 in [6] there are obtained upper and lower bounds of the convergence rate, respectively $O(w_n)$ and $O(v_n)$, which provide a near-optimal solution to the Gauss-Kuzmin-Lévy problem. Let μ be a probability measure on $\mathcal{B}_I$ such that $\mu \ll \lambda$. For any $n \in \mathbb{N}_+$, put

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$F_m^n(x) = \mu(\tau_m^n < x)$, $x \in I$, where F_m^0 is the identity map. Let

$f_m^0(x) = ((m-1)x+1)((m-1)x+m)(F^0)'(x), x \in I$

where $(F^0)' = \frac{d\mu}{d\lambda}$. Let us recall this theorem.

THEOREM 1 (Near optimal solution to Gauss-Kuzmin-Lévy) Let $f_m^0 \in C(I)$ such that $(f_m^0)' > 0$ and let μ be a probability measure on B_I such that $\mu \ll \lambda$. For any $n \in \mathbb{N}_+$ and $x \in I$ we have

$$\frac{m\alpha_m \min_{x\in I}(f_m^0)'(x)}{2\beta_m k_m^2} v_m^n G_m(x)(1-G_m(x)) \leq$$

$$\leq \left|\mu(T_m^n < x) - G_m(x)\right| \leq$$

$$\leq \frac{m(2m-1)\beta_m \max_{x\in I}(f_m^0)'(x)}{2\alpha_m k_m^2} w_m^n G_m(x)(1-G_m(x)),$$

where $G_m(x) = \frac{k_m}{(m-1)^2}\log\frac{m((m-1)x+1)}{(m-1)x+m}$.

For example, for $m=5$, the equation $E_m(x)=0$, with $e_m = \sqrt[3]{5} = 1.709975$, has as unique acceptable solution $a = a_m = 0.428487$. For this value of a, the function $\frac{\varphi_m}{V_m\varphi_m}$ attains its maximum equal to 3.349763 at x=0 and x=1 and has a minimum $m(a) = \frac{\varphi_m}{V_m\varphi_m}(0.008438) = 3.319392$. It follows that upper and lower bounds of the convergence rate are respectively $O(w_5^n)$ and $O(v_5^n)$ as $n \to \infty$, with $v_5 > 0.298528$ and $w_5 < 0.301259$.

Obviously, the determination of the exact convergence rate remains an open question. We may derive it using the same strategy as in [3].

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