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# $(h,\varphi)_{\varepsilon}-$ OPTIMALITY CONDITIONS FOR MULTI-OBJECTIVE FRACTIONAL SEMI-INFINITE PROGRAMMING WITH UNIFORM *K- (Fb,p)* CONVEXITY

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***Abstract**: Base on algebraic operation introduced by Ben Tal [A. Ben Tal, On generalized means and generalized convex functions, J. Optim. Theory Appl. 21 (1977) 1–13] and a new generalized pseudo-operation with one parameter of the following form: $x\oplus_{\varepsilon} y=h^{-1}\left(h\left(x\right)+\varepsilon h\left(y\right)\right)$,where h is an n vector-valued continuous function, defined on a subset H of $R^n$ and possessing an inverse function $h^{-1}$, $\varepsilon$ is a arbitrary but fixed positive real number, the nonsmooth generalized convex functions called uniform $K-(F_b,\rho)-$convex function, uniform $K-(F_b,\rho)-$pseudoconvex function, uniform $K-(F_b,\rho)-$quasiconvex function are defined in sense of $(h,\varphi)_{\varepsilon}$ . The nonsmooth multi-objective fractional semi-infinite programming involving these generalized convex functions is researched, and some sufficient optimality conditions are obtained.*

***Key words:** Nonsmooth, multi-objective fractional semi-infinite programming, optimality conditios, uniform $(h,\varphi)_{\varepsilon}-K-(F_b,\rho)-$convex function.*

## 1. Introduction

The convexity theory plays an important role in many aspects in mathematical programming. In recent years, to relax convexity assumption involved in sufficient conditions for optimality or duality theorems, various generalizations of convex functions have appeared in the literature.

Hanson and Mond introduced type I and type II function[5]. Reuda and Hanson extended type I function and obtained pseudo type I and quasi type I function[16]. Bector and Singh introduced *b*−convex function[1]. Bector, Suneja and Cupta extended *b*−convex function and defined univex function[2]. Mishra discussed the optimality and duality for multi-objective programming with generalized univexity[10]. Preda introduced $(F,\rho)-$convex function as extension of *F*−convex function and ρ−convex function[12-15].

Hong Yang defined $K-(F_b,\rho)-$convex function and discussed the optimality for multi-objective semi-infinite programming involving these generalized convexity[21]. As a branch of optimization, fractional programming has important practical significance in problems such as resource allocation, investment portfolio, etc. So research about fractional programming has attracted a wide spread attention. For example, Bector discussed the optimality and duality for subdifferentiable multi-objective fractional programming [3]. Liu presented three dual model of minimax fractional programming[3]. Kuk researched the optimality and duality for nonsmooth multi-objective fractional programming with generalized invexity[13]. Mishra discussed the duality for nondifferentiable minimax fractional programming involving generalized α−uniform convexity[8]. Ho researched the optimality and duality for nonsmooth minimax fractional programming involving exponential $(p,r)-$ invexity [17].Tripathy discussed the mixed type duality for multi-objective fractional programming with generalized ρ−invexity[18].

In this paper, based on a new generalized pseudo-operation with one parameter, a new class of generalized convex functions, that is, uniform $K-(F_b,\rho)-$convex function, uniform $K-(F_b,\rho)-$pseudoconvex function, uniform $K-(F_b,\rho)-$quasiconvex function are defined in sense of $(h,\varphi)_{\varepsilon}$ .We consider nonsmooth multiobjective fractional semi-infinite programming involving these generalized convex functions and obtain some sufficient optimality conditions.

## 2. Preliminaries and some definitions

Ben-Tal [4] introduced certain generalized operations of addition and multiplication.



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1) Let *h* be an *n* vector-valued continuous function, defined on a subset *H* of $R^n$ and possessing an inverse function $h^{-1}$. Define the *h*-scalar multiplication of $x \in H$ and $\lambda \in R$ as

$$\lambda \otimes x = h^{-1}\left(\lambda h(x)\right).$$

2) Let *φ* be a real-valued continuous functions, defined on $\Phi \subseteq R$ and possessing an inverse functions $\varphi^{-1}$.Then the *φ*-addition of two numbers, $\alpha \in \Phi$ and $\beta \in \Phi$, is given by

$$\alpha[+]\beta = \varphi^{-1}\left(\varphi(\alpha)+\varphi(\beta)\right),$$

and the *φ*-scalar multiplication of $\alpha \in \Phi$ and $\lambda \in R$ by

$$\lambda[\cdot]\alpha = \varphi^{-1}\left(\lambda\varphi(\alpha)\right).$$

3) The (*h, φ*)-inner product of vectors $x, y \in H$ is defined as

$$\left(x^T y\right)_{h,\varphi} = \varphi^{-1}\left(h(x)^T h(y)\right).$$

Denote

$$\left[\sum_{i=1}^{m}\right]\alpha_i = \alpha_1[+]\alpha_2[+]...[+]\alpha_m, \quad \alpha_i \in \Phi, \quad i = 1,2,...,m;$$

$$\alpha[-]\beta = \alpha[+]\left((-1)[\cdot]\beta\right).$$

By Ben-Tal generalized algebraic operation, it is easy to obtain the following conclusions[20]:

$$\left[\sum_{i=1}^{m}\right]\alpha_i = \varphi^{-1}\left(\sum_{i=1}^{m}\varphi(\alpha_i)\right)$$

$$\varphi\left(\lambda[\cdot]\alpha\right) = \lambda\varphi(\alpha)$$

$$h\left(\lambda \otimes x\right) = \lambda h(x)$$

We introduce a new pseudo-operation of addition.

4) Let $\varepsilon$ be arbitrary but fixed positive real number. Let *h* be an *n* vector-valued continuous function, defined on a subset *H* of $R^n$ and possessing an inverse function $h^{-1}$. Define the right $\varepsilon$ - *h*-vector addition of $x \in H$ and $y \in H$ as

$$x \oplus_\varepsilon y = h^{-1}\left(h(x)+\varepsilon h(y)\right)$$

Denote

$$\mathop{\oplus_\varepsilon}_{i=1}^{m} x^i = x^1 \oplus_\varepsilon x^2 \oplus_\varepsilon ... \oplus_\varepsilon x^m, \quad x^i \in H, \quad i = 1,2,...,m$$

It is easy to obtain the following conclusion:

$$\mathop{\oplus_\varepsilon}_{i=1}^{m} x^i = h^{-1}\left(h\left(x^1\right)+\varepsilon\sum_{i=2}^{m}h\left(x^i\right)\right)$$

**Lemma 2.1:** [20] *Suppose* $\varphi : R \to R$ *is a continuous one-to-one strictly monotone and onto function, and* $\alpha, \beta \in \Phi$. *Then*

$\alpha < \beta$ *if and only if* $\alpha[-]\beta < 0_\varphi$,

*where* $0_\varphi = \varphi^{-1}(0)$.

**Lema 2.2:** [23, 24]. *Let I = 1, 2, ..., m. The following statements hold:*

(1) $\lambda[\cdot]\left(\mu[\cdot]\alpha\right) = \mu[\cdot]\left(\lambda[\cdot]\alpha\right) = \lambda\mu[\cdot]\alpha$, for $\lambda, \mu, \alpha \in R$

(2) $\lambda[\cdot]\left(\alpha[-]\beta\right) = \lambda[\cdot]\alpha[-]\lambda[\cdot]\beta$, for $\lambda, \alpha, \beta \in R$

(3) $\left[\sum_{i=1}^{m}\right]\left(\alpha_i[-]\beta_i\right) = \left[\sum_{i=1}^{m}\right]\alpha_i[-]\left[\sum_{i=1}^{m}\right]\beta_i$ for $\alpha_i, \beta_i \in R$

**Lemma 2.3:** [23, 24] Suppose that function φ, which appears in Ben-Tal generalized algebraic operations, is strictly monotone with φ(0) = 0. Then, the following statements hold*:*

(1) let $\lambda \geqq 0, \alpha, \beta \in R$, and $\alpha \leqq \beta$. Then $\lambda[\cdot]\alpha \leqq \lambda[\cdot]\beta$;

(2) let $\lambda > 0, \alpha, \beta \in R$, and $\alpha < \beta$. Then $\lambda[\cdot]\alpha < \lambda[\cdot]\beta$;

(3) $\lambda < 0, \alpha, \beta \in R$, and $\alpha \leqq \beta$. Then $\lambda[\cdot]\alpha \geqq \lambda[\cdot]\beta$;

$\alpha_i, \beta_i \in R, i = 1,2,...,m$. If $\alpha_i \leqq \beta_i$ for any $i \in M$, *th en*

$$\left[\sum_{i=1}^{m}\right]\alpha_i \leqq \left[\sum_{i=1}^{m}\right]\beta_i.$$



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If $\alpha_i \leqq \beta_i$ for any *i = 1,2,...,m*, *and* there exists at least an index k such that $x_k < y_k$, then

$$\left[\sum_{i=1}^{m}\right]\alpha_i < \left[\sum_{i=1}^{m}\right]\beta_i .$$

**Lemma 2.4:** [23, 24]. Suppose that φ is a continuous one-to-one strictly monotone and onto function with φ(0) = 0. Let $\alpha, \beta \in R$. Then,

(1) $\alpha < \beta$ if and only if $\alpha[-]\beta < 0$,

(2) $\alpha[+]\beta < 0$ if and only if $\alpha = (-1)[\cdot]\beta$.

Now, we consider the following multi-objective fractional semi-infinite programming problem:

$$(VFP)\begin{cases} \min \ \dfrac{f(x)}{g(x)} = \left(\dfrac{f_1(x)}{g_1(x)}, \dfrac{f_2(x)}{g_2(x)}, \dots, \dfrac{f_p(x)}{g_p(x)}\right) \\ s.t.\ \ \mathrm{G}(x,u) \le 0,\ \ \mathrm{x} \in \mathrm{X},\ \ \mathrm{u} \in \mathrm{U}, \end{cases}$$

where $X \neq \emptyset$ is an open subset of $R^n$, $f(x) = (f_1(x), f_2(x), ..., f_p(x) : X \to R^p$, $g(x) = (g_1(x), g_2(x), ..., g_p(x) : X \to R^p$, $G : X \times U \to R$, $U \subset R$ is an infinite parameter set. $g(x) > 0,\ \forall x \in X$.

Let
$\mathrm{X}^0 = \{\mathrm{x} \,|\, G(x,u) \le 0,\ \mathrm{x} \in \mathrm{X},\ \mathrm{u} \in \mathrm{U}\}$,
$\Delta = \{\mathrm{i} \,|\, G(x,u^i) \le 0,\ \mathrm{x} \in \mathrm{X},\ \mathrm{u}^{\mathrm{i}} \in \mathrm{U}\}$
$\mathrm{I}(\overline{\mathrm{x}}) = \{\mathrm{i} \,|\, G(\overline{x},u^i) \le 0,\ \overline{\mathrm{x}} \in \mathrm{X},\ \mathrm{u}^{\mathrm{i}} \in \mathrm{U}\}$,
$\mathrm{U}_\Delta = \{\mathrm{u}^{\mathrm{i}} \,|\, G(x,u^i) \le 0,\ \mathrm{x} \in \mathrm{X},\ \mathrm{i} \in \Delta\}$ is any countable subset of *U*.
$\Lambda = \{\mu_{\mathrm{j}} \,|,\ \mu_{\mathrm{j}} \ge 0,\ \mathrm{j} \in \Delta$ there is only finite $\mu_{\mathrm{j}}$ such that $\mu_{\mathrm{j}} \neq 0\}$

*Notations.* If $\mathrm{x,y} \in R^{\mathrm{n}}$, then $x < y \iff x_i < y_i,\ i = 1, 2, ..., n$;
$x \le y \iff x_i \le y_i,\ i = 1, 2, ..., n,$ and
there exists at least one $\mathrm{i}_0 \in \{1, 2, ..., n\}$ such that $x_{i_0} < y_{i_0}$

**Definition 2.1 :** Let $K(\cdot,\cdot)$ is a local cone approximation. The function $f^K(x,\cdot) : X \to R$ with

$$(f^K(x,y))_{(h,\varphi)_\varepsilon} := \inf\{\xi \in R \,|\, (y,\xi) \in K(epif, (x, f(x)), y \in R^n)\}$$

is called $(h,\varphi)_\varepsilon - K -$directional derivative of *f* at $x$.

**Definition 2.2 :** A function $f : X \to R$ is called $(h,\varphi)_\varepsilon - K -$subdifferentiable at $x$ if there exists a convex compact set $(\partial^K f(x))_{(h,\varphi)_\varepsilon}$ such that $(f^K(x,y))_{(h,\varphi)_\varepsilon} = \max(\xi, y)_{(h,\varphi)_\varepsilon}, y \in R^n$, where

$$(\partial^K f(x))_{(h,\varphi)_\varepsilon} := \{x^* \in X \,|\, (y,x^*)_{(h,\varphi)_\varepsilon} \le (f^K(x,y))_{(h,\varphi)_\varepsilon},\ \forall y \in R^n\}$$

is called $(h,\varphi)_\varepsilon - K -$subdifferential of *f* at $x$.

**Definition 2.3 :** Let $f$ be a real-valued function defined on $R^n$, denote $\hat{f}(t) = \varphi\left(f\left(h^{-1}(t)\right)\right)$. For simplicity, write $\hat{f}(t) = \varphi f h^{-1}(t)$. The function $f$ is said to be $(h,\varphi)_\varepsilon$ -differentiable at $x$, if $\hat{f}(t)$ is differentiable at $t = h(x)$. Denote $\nabla_\varepsilon^* f(x) = h^{-1}\left(\varepsilon \nabla \hat{f}(t)_{|t=h(x)}\right)$. In addition, it is said that *f* is $(h,\varphi)_\varepsilon -$differentiable on $X \subset R^n$ if it is $(h,\varphi)_\varepsilon -$differentiable at each $x \in X$

**Definition 2.4:** A functional $F : X \times X \times R^n \to R\ (X \subset R^n)$ is called $(h,\varphi)_\varepsilon -$sublinear with respect to the third variable, if for any $x_1, x_2 \in X$,

(i) $F(x_1, x_2, a_1 \oplus_\varepsilon a_2) \le F(x_1, x_2, a_1)[+]F(x_1, x_2, \mathrm{a}_2),\ \forall a_1, a_2 \in R^n$;
(ii) $F(x_1, x_2, r \otimes a) = r[\cdot]F(x_1, x_2, a),\ \forall \mathrm{r} \in \mathrm{R},\ \mathrm{r} \ge 0,\ a \in R^n$.

**Definition 2.5:**[22] $x^* \in X^0$ is called an efficient solution for $(VFP)$ if and only if there exists no $x \in X^0$ such that $\dfrac{f(x^*)}{g(x^*)} \le \dfrac{f(x)}{g(x)}$.

**Definition 2.6:** [22] $x^* \in X^0$ is called an weak efficient solution for $(VFP)$ if and only if there exists no $x \in X^0$ such that $\dfrac{f(x^*)}{g(x^*)} < \dfrac{f(x)}{g(x)}$.



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In the following definitions, we suppose $C \subset R^n$ is a nonempty set, $x_0 \in C$, $f: C \to R$ is a local Lipschitz function at $x_0$, $F: C \times C \times R^n \to R$ is $(h,\varphi)_\varepsilon -$sublinear with respect to the third variable, $\phi: R \to R$, $b: C \times C \times [0,1] \to R_+$, $\lim_{\lambda \to 0^+} b(x,x_0) = b(x,x_0)$, $d(\cdot,\cdot)$ is a pseudo-metric in $R^n$.

In [6], Elster and Thier-Felder defined $K-$directional derivative and $K-$subdifferential and pointed out that $K-$subdifferential is most generalized. In [9], using directional $K-$derivative and $K-$subdifferential, some new generalized convex functions are defined. In [22] Hong Yang and Yongchun He defined $K-(F_b,\rho)-$convexity, $K-(F_b,\rho)-$pseudoconvexity, $K-(F_b,\rho)-$quasiconvexity. Using these definitions we now define:

**Definition 2.7:** Let $\rho \in R$. A function $f: C \to R$ is said to be uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$convex at $x_0$ with respect to $F,\phi,b,d$ if for all $x \in C$, we have

$$b(x,x_0)[\cdot]\phi\left(f(x)[-]f(x_0)\right) \geq F\left(x,x_0,\nabla_\varepsilon^* f(x_0)\right)[+]\rho[\cdot]d^2(x,x_0).$$

**Definition 2.8:** Let $\rho \in R$. A function $f: C \to R$ is said to be strictly uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$convex at $x_0$ with respect to $F,\phi,b,d$ if for all $x \in C$, $x \neq x_0$ we have

$$b(x,x_0)[\cdot]\phi\left(f(x)[-]f(x_0)\right) > F\left(x,x_0,\nabla_\varepsilon^* f(x_0)\right)[+]\rho[\cdot]d^2(x,x_0).$$

**Definition 9:** Let $\rho \in R$. A function $f: C \to R$ is said to be uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$pseudoconvex at $x_0$ with respect to $F,\phi,b,d$ if for all $x \in C$, we have

$$b(x,x_0)[\cdot]\phi\left(f(x)[-]f(x_0)\right) < 0 \Rightarrow F\left(x,x_0,\nabla_\varepsilon^* f(x_0)\right)[+]\rho[\cdot]d^2(x,x_0) < 0.$$

**Definition 10:** Let $\rho \in R$. A function $f: C \to R$ is said to be strictly uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$pseudoconvex at $x_0$ with respect to $F,\phi,b,d$ if for all $x \in C$, $x \neq x_0$, we have

$$b(x,x_0)[\cdot]\phi\left(f(x)[-]f(x_0)\right) \leq 0 \Rightarrow F\left(x,x_0,\nabla_\varepsilon^* f(x_0)\right)[+]\rho[\cdot]d^2(x,x_0) < 0.$$

**Definition 11:** Let $\rho \in R$. A function $f: C \to R$ is said to be uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$quasiconvex at $x_0$ with respect to $F,\phi,b,d$ if for all $x \in C$, we have

$$b(x,x_0)[\cdot]\phi\left(f(x)[-]f(x_0)\right) \leq 0 \Rightarrow F\left(x,x_0,\nabla_\varepsilon^* f(x_0)\right)[+]\rho[\cdot]d^2(x,x_0) \leq 0.$$

**Definition 12:** Let $\rho \in R$. A function $f: C \to R$ is said to be weak uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$quasiconvex at $x_0$ with respect to $F,\phi,b,d$ if for all $x \in C$, we have

$$b(x,x_0)[\cdot]\phi\left(f(x)[-]f(x_0)\right) < 0 \Rightarrow F\left(x,x_0,\nabla_\varepsilon^* f(x_0)\right)[+]\rho[\cdot]d^2(x,x_0) \leq 0.$$

## 3. Sufficient $(h,\varphi)_\varepsilon$-optimality conditions

Through the rest of this paper, one further assumes that *h* is a continuous one-to-one and onto function with *h*(0) = 0. Similarly, suppose that *φ* is a continuous one-to-one strictly monotone and onto function, with *φ*(0) = 0. Under the above assumptions, it is clear that $0[\cdot]\alpha = \alpha[\cdot]0 = 0$. Also, assume that *φ* is a homogeneous function.

In this section,we obtain some suffincient conditions for a feasible $\bar{x}$ to be efficient or weak efficient for $(VFP)$ in the form of the following theorems.

**Theorem 3.1:** Assume that $\bar{x} \in X^0$, if for any $x \in X^0$, there exist $F,\phi_1,\phi_2,b_1,b_2$, $\rho_1^i \in R, \rho_2^j \in R, \lambda_i > 0,\ i = 1,2,...,p,\ \mu_j \in \Lambda$, $j \in I(\bar{x})$, such that

$(i)$ $A_i(x) = f_i(x)[-]\dfrac{f_i(\bar{x})}{g_i(\bar{x})}[\cdot]g_i(x)$ is uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$convex at $\bar{x}$, $i = 1,2,...,p$;



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$(ii)$ $G(x,u^j)$ is uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$ convex at $\bar{x}$, $j \in I(\bar{x})$;

$(iii)$ $0 = \bigoplus_{i=1}^{p}{}_{\varepsilon}\, \lambda_i \otimes \nabla_\varepsilon^* f_i(\bar{x}) \oplus_\varepsilon \bigoplus_{j\in I(\bar{x})}{}_{\varepsilon}\, \mu_j \otimes \nabla_\varepsilon^* G(x,u^j)$, $\forall u^j \in U_\Delta$;

$(iv)$ $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0,\ \phi_1(0)=0,$
$\alpha \le 0 \Rightarrow \phi_2(\alpha) \le 0,$ ;
$b_1(x,\bar{x}) > 0,\ b_2(x,\bar{x}) \ge 0;$

$(v)$ $\sum_{i=1}^{p} \lambda_i \rho_1^i + \sum_{j\in I(\bar{x})} \mu_j \rho_2^j \ge 0$.

Then $\bar{x}$ is an efficient solution for $(VFP)$.

Proof: By hypothesis (iii), we have

$$0 = \bigoplus_{i=1}^{p}{}_{\varepsilon}\, \lambda_i \otimes \nabla_\varepsilon^* f_i(\bar{x}) \oplus_\varepsilon \bigoplus_{j\in I(\bar{x})}{}_{\varepsilon}\, \mu_j \otimes \nabla_\varepsilon^* G(x,u^j)$$

So

$$F(x,\bar{x},\bigoplus_{i=1}^{p}{}_{\varepsilon}\, \lambda_i \otimes \nabla_\varepsilon^* f_i(\bar{x}) \oplus_\varepsilon \bigoplus_{j\in I(\bar{x})}{}_{\varepsilon}\, \mu_j \otimes \nabla_\varepsilon^* G(x,u^j)) = = F(x,\bar{x},0) = 0 \qquad (1)$$

Suppose that $\bar{x}$ is not an efficient solution for $(VFP)$, then there exists $x \in X^0$ and at least one $i_0 \in \{1,2,...,p\}$ such that

$$\frac{f_{i_0}(x)}{g_{i_0}(x)} < \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},$$

$$\frac{f_i(x)}{g_i(x)} \le \frac{f_i(\bar{x})}{g_i(\bar{x})},\ i=1,2,...,p,\ i \ne i_0.$$

Thus

$$f_{i_0}(x) < \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})} g_{i_0}(x) \qquad (2)$$

Without loss of generality, we suppose $\varphi$ is strictly monotone increasing on $R$.

Applying $\varphi$ in relation (2) we obtain

$$\varphi(f_{i_0}(x)) < \varphi\left(\frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})} g_{i_0}(x)\right)$$

and because $\varphi$ is a homogeneous function, we have

$$\varphi(f_{i_0}(x)) < \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})} \varphi(g_{i_0}(x))$$

So

$$\varphi(f_{i_0}(x)) - \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})} \varphi(g_{i_0}(x)) < 0$$

Applying $\varphi^{-1}$ in this relation, we get

$$\varphi^{-1}\left(\varphi(f_{i_0}(x)) - \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})} \varphi(g_{i_0}(x))\right) < \varphi^{-1}(0) = 0$$

Thus

$$A_{i_0}(x) = f_{i_0}(x)[-]\frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})}[\cdot] g_{i_0}(x) < 0 = A_{i_0}(\bar{x}).$$

In the same way it can demonstrate that

$$A_i(x) = f_i(x)[-]\frac{f_i(\bar{x})}{g_i(\bar{x})}[\cdot] g_i(x) < 0 = A_i(\bar{x}),$$
$$i=1,2,...,p,\ i \ne i_0$$

By hypothesis (iv), lema 2.3 and assuming that $\varphi$ is strictly monotone increasing on $R$, we obtain

$$b_1(x,\bar{x})[\cdot]\phi_1\left(A_{i_0}(x)[-]A_{i_0}(\bar{x})\right) < 0$$

$$b_1(x,\bar{x})[\cdot]\phi_1\left(A_i(x)[-]A_i(\bar{x})\right) \le 0,$$
$$i=1,2,...,p,\ i \ne i_0$$

From hypothesis (i) we get

$$F(x,\bar{x},\nabla_\varepsilon^* A_{i_0}(\bar{x}))[+]\rho_1^{i_0}[\cdot]d^2(x,\bar{x}) < 0$$

$$F(x,\bar{x},\nabla_\varepsilon^* A_i(\bar{x}))[+]\rho_1^{i}[\cdot]d^2(x,\bar{x}) \le 0,$$
$$i=1,2,...,p,\ i \ne i_0$$



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and, since $\lambda_i > 0,\ i = 1,2,...,p$, together with lema 2.3, we have

$$\left[\sum_{i=1}^{p}\right]\lambda_i[\cdot]F(x,\overline{x},\nabla_\varepsilon{}^{*}A_i(\overline{x}))[+]\left[\sum_{i=1}^{p}\right]\lambda_i\rho_1^i[\cdot]d^2(x,\overline{x})<0 \quad (3)$$

Observing that $G(x,u^j) \le 0 = G(\overline{x},u^j),\ j \in I(\overline{x})$, we have

$$G(x,u^j)[-]G(\overline{x},u^j) \le 0,\ j \in I(\overline{x}).$$

and by hypothesis (iv), we have

$$b_2(x,\overline{x})[\cdot]\phi_2\left(h^*(x,u^j)[-]h^*(\overline{x},u^j)\right) \le 0,$$
$$j \in I(\overline{x})$$

By hypothesis (ii), we get

$$F(x,\overline{x},\nabla_\varepsilon{}^{*}G(\overline{x},u^j))[+]\rho_2^j[\cdot]d^2(x,\overline{x}) \le 0,$$
$$j \in I(\overline{x})$$

Since $\mu_j \in \Lambda,\ j \in I(\overline{x})$, we have

$$\left[\sum_{j\in I(\overline{x})}\right]\mu_j[\cdot]F(x,\overline{x},\nabla_\varepsilon{}^{*}G(\overline{x},u^j))[+]\left[\sum_{j\in I(\overline{x})}\right]\mu_j\rho_2^j[\cdot]d^2(x,\overline{x}) \le 0 \quad (4)$$

Adding (3) and (4) ,using the sublinearity of $F$, we can obtain

$$F(x,\overline{x},\bigoplus_{\varepsilon\, i=1}^{p}\lambda_i\otimes\nabla_\varepsilon{}^{*}A_i(\overline{x})\oplus_\varepsilon\bigoplus_{\varepsilon\, j\in I(\overline{x})}\mu_j\otimes\nabla_\varepsilon{}^{*}G(\overline{x},u^j))[+]$$
$$[+]\left(\sum_{i=1}^{p}\lambda_i\rho_1^i+\sum_{j\in I(\overline{x})}\mu_j\rho_2^j\right)[\cdot]d^2(x,\overline{x})<0.$$

By hypothesis (v), we have

$$\sum_{i=1}^{p}\lambda_i\rho_1^i+\sum_{j\in I(\overline{x})}\mu_j\rho_2^j \ge 0$$

so

$$F(x,\overline{x},\bigoplus_{\varepsilon\, i=1}^{p}\lambda_i\otimes\nabla_\varepsilon{}^{*}A_i(\overline{x})\oplus_\varepsilon\bigoplus_{\varepsilon\, j\in I(\overline{x})}\mu_j\otimes\nabla_\varepsilon{}^{*}G(\overline{x},u^j))<0$$

which contradicts (i). Therefore, $\overline{x}$ is an efficient solution for $(VFP)$.

**Theorem 3.2:** Assume that $\overline{x} \in X^0$, if for any $x \in X^0$, there exist $F,\phi_1,\phi_2,b_1,b_2$, $\rho_1^i \in R, \rho_2^j \in R, \lambda_i > 0,\ i=1,2,...,p,\ \mu_j \in \Lambda,\ j \in I(\overline{x})$ such that

$(i)$ $A_i(x) = f_i(x)[-]\dfrac{f_i(\overline{x})}{g_i(\overline{x})}[\cdot]g_i(x)$ is uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$ convex at $\overline{x}$, $i=1,2,...,p$;

$(ii)$ $G(x,u^j)$ is uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$ convex at $\overline{x}$, $j \in I(\overline{x})$;

$(iii)$ $0 = \bigoplus_{\varepsilon\, i=1}^{p}\lambda_i\otimes\nabla_\varepsilon{}^{*}f_i(\overline{x})\oplus_\varepsilon\bigoplus_{\varepsilon\, j\in I(\overline{x})}\mu_j\otimes\nabla_\varepsilon{}^{*}G(x,u^j),$ $\forall u^j \in U_\Delta$;

$(iv)$ $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0,\ \phi_1(0) = 0,$
$\alpha \le 0 \Rightarrow \phi_2(\alpha) \le 0,$
$b_1(x,\overline{x}) > 0,\ b_2(x,\overline{x}) \ge 0;$

$(v)$ $\sum_{i=1}^{p}\lambda_i\rho_1^i+\sum_{j\in I(\overline{x})}\mu_j\rho_2^j \ge 0$.

Then $\overline{x}$ is a weak efficient solution for $(VFP)$.

**Theorem 3.3:** Assume that $\overline{x} \in X^0$, if for any $x \in X^0$, there exist $F,\phi_1,\phi_2,b_1,b_2$, $\rho_1^i \in R, \rho_2^j \in R, \lambda_i > 0,\ i=1,2,...,p,\ \mu_j \in \Lambda,\ j \in I(\overline{x})$ such that

$(i)$ $A_i(x) = f_i(x)[-]\dfrac{f_i(\overline{x})}{g_i(\overline{x})}[\cdot]g_i(x)$ is strictly uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$ convex at $\overline{x}$, $i=1,2,...,p$;

$(ii)$ $\mathrm{G}(x,u^j)$ is uniform $(h,\varphi)_\varepsilon - K - (F_b,\rho)-$ quasiconvex at $\overline{x}$, $j \in I(\overline{x})$;

$(iii)$ $0 = \bigoplus_{\varepsilon\, i=1}^{p}\lambda_i\otimes\nabla_\varepsilon{}^{*}f_i(\overline{x})\oplus_\varepsilon\bigoplus_{\varepsilon\, j\in I(\overline{x})}\mu_j\otimes\nabla_\varepsilon{}^{*}G(x,u^j),$ $\forall u^j \in U_\Delta$;

$(iv)$ $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0,\ \phi_1(0) = 0,$
$\alpha \le 0 \Rightarrow \phi_2(\alpha) \le 0,$
$b_1(x,\overline{x}) > 0,\ b_2(x,\overline{x}) \ge 0;$

$(v)$ $\sum_{i=1}^{p}\lambda_i\rho_1^i+\sum_{j\in I(\overline{x})}\mu_j\rho_2^j \ge 0$.

Then $\overline{x}$ is an efficient solution for $(VFP)$.



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**Theorem 3.4:** Assume that $\overline{x} \in X^0$, if for any $x \in X^0$, there exist $F, \phi_1, \phi_2, b_1, b_2$, $\rho_1^i \in R, \rho_2^j \in R, \lambda_i > 0,\ i = 1, 2, ..., p,\ \mu_j \in \Lambda$, $j \in I(\overline{x})$ such that

$(i)$ $A_i(x) = f_i(x)[-]\dfrac{f_i(\overline{x})}{g_i(\overline{x})}[\cdot]g_i(x)$ is uniform $(h,\varphi)_\varepsilon - K - (F_b, \rho) -$ pseudoconvex at $\overline{x}$, $i = 1, 2, ..., p$;

$(ii)$ $G(x, u^j)$ is uniform $(h,\varphi)_\varepsilon - K - (F_b, \rho) -$ quasiconvex at $\overline{x}$, $j \in I(\overline{x})$;

$(iii)$ $0 = \bigoplus_{\varepsilon\, i=1}^{p} \lambda_i \otimes \nabla_\varepsilon^* f_i(\overline{x}) \oplus_\varepsilon \bigoplus_{\varepsilon\, j\in I(\overline{x})} \mu_j \otimes \nabla_\varepsilon^* G(x, u^j)$, $\forall u^j \in U_\Delta$;

$(iv)$ $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0,\ \phi_1(0) = 0$
$\alpha \le 0 \Rightarrow \phi_2(\alpha) \le 0$,
$b_1(x, \overline{x}) > 0,\ b_2(x, \overline{x}) \ge 0$;

$(v)$ $\sum_{i=1}^{p} \lambda_i \rho_1^i + \sum_{j\in I(\overline{x})} \mu_j \rho_2^j \ge 0$.

Then $\overline{x}$ is a weak efficient solution for $(VFP)$.

**Theorem 3.5:** Assume that $\overline{x} \in X^0$, if for any $x \in X^0$, there exist $F, \phi_1, \phi_2, b_1, b_2$, $\rho_1^i \in R, \rho_2^j \in R, \lambda_i \ge 0,\ i = 1, 2, ..., p,\ \mu_j \in \Lambda$, and not all $\mu_j$ are zero, $j \in I(\overline{x})$, such that

$(i)$ $A_i(x) = f_i(x)[-]\dfrac{f_i(\overline{x})}{g_i(\overline{x})}[\cdot]g_i(x)$ is uniform $(h,\varphi)_\varepsilon - K - (F_b, \rho) -$ quasiconvex at $\overline{x}$, $i = 1, 2, ..., p$;

$(ii)$ $G(x, u^j)$ is strictly uniform $(h,\varphi)_\varepsilon - K - (F_b, \rho) -$ pseudoconvex at $\overline{x}$, $j \in I(\overline{x})$;

$(iii)$ $0 = \bigoplus_{\varepsilon\, i=1}^{p} \lambda_i \otimes \nabla_\varepsilon^* f_i(\overline{x}) \oplus_\varepsilon \bigoplus_{\varepsilon\, j\in I(\overline{x})} \mu_j \otimes \nabla_\varepsilon^* G(x, u^j)$, $\forall u^j \in U_\Delta$;

$(iv)$ $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0,\ \phi_1(0) = 0$,
$\alpha \le 0 \Rightarrow \phi_2(\alpha) \le 0$,
$b_1(x, \overline{x}) \ge 0,\ b_2(x, \overline{x}) \ge 0$;

$(v)$ $\sum_{i=1}^{p} \lambda_i \rho_1^i + \sum_{j\in I(\overline{x})} \mu_j \rho_2^j \ge 0$.

Then $\overline{x}$ is an efficient solution for $(VFP)$.

**Theorem 3.6:** Assume that $\overline{x} \in X^0$, if for any $x \in X^0$, there exist $F, \phi_1, \phi_2, b_1, b_2$, $\rho_1^i \in R, \rho_2^j \in R, \lambda_i \ge 0,\ i = 1, 2, ..., p,\ \mu_j \in \Lambda$, and not all $\mu_j$ are zero, $j \in I(\overline{x})$, such that

$(i)$ $A_i(x) = f_i(x)[-]\dfrac{f_i(\overline{x})}{g_i(\overline{x})}[\cdot]g_i(x)$ is weak uniform $(h,\varphi)_\varepsilon - K - (F_b, \rho) -$ quasiconvex at $\overline{x}$, $i = 1, 2, ..., p$;

$(ii)$ $G(x, u^j)$ is strictly uniform $(h,\varphi)_\varepsilon - K - (F_b, \rho) -$ pseudoconvex at $\overline{x}$, $j \in I(\overline{x})$;

$(iii)$ $0 = \bigoplus_{\varepsilon\, i=1}^{p} \lambda_i \otimes \nabla_\varepsilon^* f_i(\overline{x}) \oplus_\varepsilon \bigoplus_{\varepsilon\, j\in I(\overline{x})} \mu_j \otimes \nabla_\varepsilon^* G(x, u^j)$, $\forall u^j \in U_\Delta$;

$(iv)$ $\alpha < 0 \Rightarrow \phi_1(\alpha) < 0,\ \phi_1(0) = 0$,
$\alpha \le 0 \Rightarrow \phi_2(\alpha) \le 0$,
$b_1(x, \overline{x}) \ge 0,\ b_2(x, \overline{x}) \ge 0$;

$(v)$ $\sum_{i=1}^{p} \lambda_i \rho_1^i + \sum_{j\in I(\overline{x})} \mu_j \rho_2^j \ge 0$.

Then $\overline{x}$ is a weak efficient solution for $(VFP)$.

The proofs of Theorem 2—Theorem 6 are similar to Theorem 1.

## CONCLUSIONS

In this paper, we consider nonsmooth multi-objective fractional semi-infinite programming involving a new pseudo-operation and a new classes of generalized convex functions, that is, uniform $(h,\varphi)_\varepsilon - K - (F_b, \rho) -$ convex function, uniform $(h,\varphi)_\varepsilon - K - (F_b, \rho) -$ pseudoconvex function, uniform $(h,\varphi)_\varepsilon - K - (F_b, \rho) -$ quasiconvex function and obtain some sufficient optimality conditions.

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