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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN COMPLEX VALUED METRIC SPACE

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ABSTRACT. In this paper, using the (CLR) and $(E.A)$ properties of the involved pairs, common fixed point results for four and six weakly compatible self-mappings are established in complex valued metric spaces. Our results include some known results as special cases.

1. Introduction

Azam *et al.* [**2**] introduced the notion of complex valued metric space which is more general than ordinary metric space and studied common fixed point theorems for mappings satisfying a rational type inequality. Verma and Pathak [**16**] introduced the concept of property $(E.A)$ and (CLR) property in a complex valued metric space and proved some common fixed point theorems for two pairs of weakly compatible self-mappings, satisfying a contractive condition of maximum type. Kumar *et al.* [**8**] and Ozturk [**10**] established common fixed point theorems for two pairs of weakly compatible mappings in complex valued metric spaces. Several authors [**11, 15, 12**] proved common fixed point theorem with six self-maps in the context of complex valued metric spaces.

The aim of this manuscript is to prove common fixed point theorems for two pairs of weakly compatible mappings, satisfying contractive condition of rational type using property $(E.A)$ and (CLR) property in complex valued metric spaces. Furthermore, we establish common fixed point theorems for three pairs of weakly compatible mappings in complex valued metric spaces. Our results generalizes the results of [**8, 10**] in complex valued metric spaces.

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To proceed further, we recollect some known definitions and results from the literature which are helpful for proving our main result.

2. Preliminaries

Definition 2.1. ([**2**]) *Let $\mathbb{C}$ be the set of complex numbers and $z, w \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:*
$z \precsim w$ if and only if $Re(z) \leqslant Re(w)$ and $Im(z) \leqslant Im(w)$,
$z \prec w$ if and only if $Re(z) < Re(w)$ and $Im(z) < Im(w)$.
Note that

i) $k_1, k_2 \in \mathbb{R}$ *and* $k_1 \leqslant k_2 \Rightarrow k_1 z \precsim k_2 z$, *for all* $z \in \mathbb{C}$.
ii) $0 \precsim z \precnsim w \Rightarrow |z| < |w|$, *for all* $z, w \in \mathbb{C}$.
iii) $z \precsim w$ *and* $w \prec w^* \Rightarrow z \prec w^*$, *for all* $z, w, w^* \in \mathbb{C}$.

Definition 2.2. ([**16**]) *The "* max *" function for the partial order relation "* $\precsim$ *" on $\mathbb{C}$ is defined by the following way: for all $z_1, z_2, z_3 \in \mathbb{C}$,*

1) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$;
2) *If* $z_1 \precsim \max\{z_2, z_3\}$, *then* $z_1 \precsim z_2$ *or* $z_1 \precsim z_3$;
3) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ *or* $|z_1| \leqslant |z_2|$.

Definition 2.3. ([**2, 14**]) *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfying the following axioms:*

1) $0 \precsim d(z_1, z_2)$, *for all* $z_1, z_2 \in X$ *and* $d(z_1, z_2) = 0$ *if and only if* $z_1 = z_2$;
2) $d(z_1, z_2) = d(z_2, z_1)$, *for all* $z_1, z_2 \in X$;
3) $d(z_1, z_2) \precsim d(z_1, z_3) + d(z_3, z_2)$, *for all* $z_1, z_2, z_3 \in X$.

Then the pair (X, d) is called a complex valued metric space.

Definition 2.4. ([**2**]) *Let $\{z_r\}$ be a sequence in complex valued metric (X, d) and $z \in X$. Then z is called the limit of $\{z_r\}$ if for every $w \in \mathbb{C}$, with $0 \prec w$, there is $r_0 \in \mathbb{N}$, such that $d(z_r, z) \prec w$ for all $r > r_0$ and we write $\lim_{r\to\infty} z_r = z$.*

Lemma 2.1. *Let (X, d) be a complex valued metric space. Then a sequence $\{z_r\}$ in X converges to z if and only if $|d(z_r, z)| \to 0$ as $r \to \infty$.*

Definition 2.5 ([**3, 13**]). *Let S and T be two self-maps on a non-empty set X. Then*

i) $z \in X$ *is called a fixed point of* S *if* $Sz = z$.
ii) $z \in X$ *is called a coincidence point of* S *and* T *if* $Sz = Tz$.
iii) $z \in X$ *is called a common fixed point of* S *and* T *if* $Sz = Tz = z$.

Definition 2.6 ([**4, 7**]). *Let (X, d) be a complex valued metric space. Then a pair of mappings $S, T : X \to X$ is weakly compatible if they commute at their coincidence points, that is, if there exist a point $z \in X$ such that $STz = TSz$ whenever $Sz = Tz$.*

Definition 2.7 ([**1, 16**]). *Let $S, T : X \to X$ be two self-maps on a complex-valued metric space (X, d). Then the pair (S, T) is said to satisfy property $(E.A)$, if there exists a sequence $\{z_n\}$ in X such that*

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = z \quad \text{for some} \quad z \in X.$$

DEFINITION 2.8 ([**9, 5**]). *Let* (X,d) *be a complex valued matric space and* $A,B,S,T : X \to X$ *be four self-maps. Then the pairs* (A,S) *and* (B,T) *satisfy the common* $(E.A)$ *property if there exist two sequences* $\{z_n\}$ *and* $\{w_n\}$ *in* X *such that*

$$\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Bw_n = \lim_{n\to\infty} Tw_n = z \in X.$$

EXAMPLE 2.1 ([**11**]). *Let* (X,d) *be a complex valued metric space where* $X = \mathbb{C}$. *Define* $A,B,S,T : X \to X$ *by*

$$Az = 2 - iz\ ,\ Bz = i - 2z^2\ ,\ Sz = i - 2z\ ,\ Tz = 2 + (z-2i)^3.$$

Let $\{z_n\} = \{-1 + \frac{i}{n}\}_{n\geqslant 1}$ *and* $\{w_n\} = \{\frac{1}{n} + i\}_{n\geqslant 1}$ *be the two sequences in* X. *Then*

$$\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Bw_n = \lim_{n\to\infty} Tw_n = 2 + i \in X$$

Hence the pairs (A,S) *and* (B,T) *satisfy common* $(E.A)$ *property.*

DEFINITION 2.9. ([**16, 6**]) *Let* $S,T : X \to X$ *be two self-maps on a complex-valued metric space* (X,d). *Then* S *and* T *are said to satisfy the common limit in the range of* S *property, denoted by* (CLR_S) *if there exists a sequence* $\{z_n\}$ *in* X *such that*

$$\lim_{n\to\infty} Tz_n = \lim_{n\to\infty} Sz_n = Sz \quad \text{for some} \quad z \in X.$$

DEFINITION 2.10. *Let* (X,d) *be a complex valued matric space and* $A,B,S,T : X \to X$ *be four self maps. The pairs* (A,S) *and* (B,T) *satisfy the common limit range property with respect to mapping* S *and* T, *denoted by* (CLR_{ST}) *if there exist two sequences* $\{z_n\}$ *and* $\{w_n\}$ *in* X *such that*

$$\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Bw_n = \lim_{n\to\infty} Tw_n = z \in S(X) \cap T(X).$$

3. Main Results

THEOREM 3.1. *Let* (X,d) *be a complex valued metric space and* $K,L,N,M : X \to X$ *be four self-mappings satisfying the following conditions:*

I *either the pair* (K,N) *satisfies* (CLR_K) *property or the pair* (L,M) *satisfies* (CLR_L) *property such that* $K(X) \subseteq M(X)$ *and* $L(X) \subseteq N(X)$;

II

$$\begin{aligned} d(Kx,Ly) \precsim\ & \lambda_1 d(My,Ly)\frac{1+d(Nx,Kx)}{1+d(Nx,My)} + \lambda_2 d(Nx,Kx)\frac{1+d(My,Ly)}{1+d(Nx,My)} \\ & + \lambda_3 d(Nx,Kx)\frac{1+d(Nx,Ly)+d(My,Kx)}{1+d(Nx,Kx)+d(My,Ly)} \\ & + \lambda_4 max\Big\{d(Nx,My), d(Nx,Kx), d(My,Ly)\Big\}, \end{aligned}$$

where $x,y \in X$ *and* $\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in [0,1)$ *such that* $\lambda_1+\lambda_2+\lambda_3+\lambda_4 < 1$. *If the pairs* (K,N) *and* (L,M) *are weakly compatible, then* K,L,M *and* N *have unique common fixed point in* X.

Proof. Let the pair (K,N) satisfies (CLR_K) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Nx_n = Kz \ \text{ for some } \ z \in X. \tag{3.1}$$

Since $K(X) \subseteq M(X)$, so there exists $r \in X$ such that $Kz = Mr$ and thus (3.1) becomes

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Nx_n = \lim_{n\to\infty} My_n = z. \tag{3.2}$$

Now, we assert that $\lim_{n\to\infty} Ly_n = z$. Suppose that $\lim_{n\to\infty} Ly_n = w \neq z$, then using condition (II) of Theorem 3.1 with setting $x = x_n$ and $y = y_n$, it follows that

$$\begin{aligned} d(Kx_n, Ly_n) \precsim\ & \lambda_1 d(My_n, Ly_n)\frac{1+d(Nx_n,Kx_n)}{1+d(Nx_n,My_n)} + \lambda_2 d(Nx_n,Kx_n)\frac{1+d(My_n,Ly_n)}{1+d(Nx_n,My_n)} \\ & + \lambda_3 d(Nx_n,Kx_n)\frac{1+d(Nx_n,Ly_n)+d(My_n,Kx_n)}{1+d(Nx_n,Kx_n)+d(My_n,Ly_n)} \\ & + \lambda_4 max_n\Big\{d(Nx_n,My_n), d(Nx_n,Kx_n), d(My_n,Ly_n)\Big\}. \end{aligned}$$

Taking limit as $n \to \infty$ and using (3.2), we get

$$\begin{aligned} d(z,w) \precsim \lambda_1 m_1 d(z,w) + \lambda_4 d(z,w) \quad &\Rightarrow \quad (1-\lambda_1-\lambda_4)d(z,w) \precsim 0 \\ &\Rightarrow \quad |(1-\lambda_1-\lambda_4)d(z,w)| \leqslant 0, \end{aligned}$$

but $1-\lambda_1-\lambda_4 > 0$ so that $|d(z,w)| \leqslant 0$.

Thus $z = w$ and $\lim_{n\to\infty} Ly_n = z$. Hence in view of equation (3.2), we get

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Nx_n = \lim_{n\to\infty} Ly_n = \lim_{n\to\infty} My_n = z. \tag{3.3}$$

Further, since $M(X)$ is closed subspace of X, so there exists $r \in X$ such that $Mr = z$ and it follows from (3.3) that

$$\lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Nx_n = \lim_{n\to\infty} Ly_n = \lim_{n\to\infty} My_n = z = Mr. \tag{3.4}$$

Now, we claim that $Lr = Mr$. To support the claim, let $Lr \neq Mr$. For this, setting $x = x_n$ and $y = r$ in condition (II) of Theorem, we have

$$\begin{aligned} d(Kx_n, Lr) \precsim\ & \lambda_1 d(Mr, Lr)\frac{1+d(Nx_n,Kx_n)}{1+d(Nx_n,Mr)} + \lambda_2 d(Nx_n,Kx_n)\frac{1+d(Mr,Lr)}{1+d(Nx_n,Mr)} \\ & + \lambda_3 d(Nx_n,Kx_n)\frac{1+d(Nx_n,Lr)+d(Mr,Kx_n)}{1+d(Nx_n,Kx_n)+d(Mr,Lr)} \\ & + \lambda_4 max\Big\{d(Nx_n,Mr), d(Nx_n,Kx_n), d(Mr,Lr)\Big\}. \end{aligned}$$

Taking limit as $n \to \infty$ and using (3.4), we get

$$d(z,Lr) \precsim \lambda_1 d(z,Lr) + \lambda_4 d(Lr,z) \quad \Rightarrow \quad (1-\lambda_1-\lambda_4)d(z,Lr) \precsim 0$$

But $1-\lambda_1-\lambda_5>0$, thus $d(z,w)\precsim 0$, which is possible only if $d(z,w)=0$ and hence

$$Lr = Mr = z. \tag{3.5}$$

Also, since $L(X)\subseteq N(X)$, so there exists $s\in X$ such that $Lr=Ns$ and from (3.5), we get

$$Lr = Mr = Ns = z. \tag{3.6}$$

We announce that $Ks=Ns$. For this, take $x=s$ and $y=r$ in condition (II), we have

$$\begin{aligned} d(Ks,Lr) \precsim\ & \lambda_1 d(Mr,Lr)\frac{1+d(Ns,Ks)}{1+d(Ns,Mr)} + \lambda_2 d(Ns,Ks)\frac{1+d(Mr,Lr)}{1+d(Ns,Mr)} \\ & + \lambda_3 d(Ns,Ks)\frac{1+d(Ns,Lr)+d(Mr,Ks)}{1+d(Ns,Ks)+d(Mr,Lr)} \\ & + \lambda_4 mas\Big\{d(Ns,Mr),d(Ns,Ks),d(Mr,Lr)\Big\}. \end{aligned}$$

Using equation (3.6), we can write

$$d(Ks,z)\precsim \lambda_2 d(z,Ks)+\lambda_3 d(z,Ks)+\lambda_4 d(z,Ks)$$

$$\Rightarrow\ (1-\lambda_2-\lambda_3-\lambda_4)d(Ks,z)\precsim 0 \quad\Rightarrow\quad (Ks,z)\precsim 0, \quad \text{as}\ \ 1-\lambda_2-\lambda_3-\lambda_4>0.$$

Thus $Ks=Ns$ and hence from equation (3.6) it follows that

$$Ks = Lr = Mr = Ns = z. \tag{3.7}$$

Now, using the weak compatibility of the pairs (K,N), (L,M) and equation (3.7), we have

$$Ks = Ns \quad\Rightarrow NKs = KNs \quad\Rightarrow\quad Kz = Nz. \tag{3.8}$$

and

$$Lr = Mr \quad\Rightarrow MLr = LMr \quad\Rightarrow\quad Lz = Mz. \tag{3.9}$$

Let $Kz=z$. If $Kz\neq z$, then condition (II) of Theorem 3.1 with $x=z$ and $y=r$, we have

$$\begin{aligned} d(Kz,Lr) \precsim\ & \lambda_1 d(Mr,Lr)\frac{1+d(Nz,Kz)}{1+d(Nz,Mr)} + \lambda_2 d(Nz,Kz)\frac{1+d(Mr,Lr)}{1+d(Nz,Mr)} \\ & + \lambda_3 d(Nz,Kz)\frac{1+d(Nz,Lr)+d(Mr,Kz)}{1+d(Nz,Kz)+d(Mr,Lr)} \\ & + \lambda_4 maz\Big\{d(Nz,Mr),d(Nz,Kz),d(Mr,Lr)\Big\}, \end{aligned}$$

with the help of (3.7) and (3.8), one can write $d(Kz,z)\precsim \lambda_4 d(Kz,z)$, which is contradiction. Thus $Kz=z$ and from (3.8), we get

$$Kz = Nz = z. \tag{3.10}$$

Similarly, by taking $x = s$, $y = z$ in condition (II) and using equations (3.7) and (3.9), we can easily show that

$$Lz = Mz = z. \tag{3.11}$$

Therefore from (3.10) and (3.11), we get

$$Kz = Lz = Mz = Nz = z. \tag{3.12}$$

That is, z is the common fixed point of K, L, M and N.

Similar argument arises if we assume that the pair (L, M) satisfies (CLR_L) property.

Finally, we have to show that z is the unique common fixed point of K, L, M and N. For this, assume that $z^* \neq z$ be another common fixed point of K, L, M and N. Then on using condition (II) with setting $x = z$ and $y = z^*$, we have

$$\begin{aligned} d(Kz, Lz^*) \precsim\ & \lambda_1 d(Mz^*, Lz^*)\frac{1 + d(Nz, Kz)}{1 + d(Nz, Mz^*)} + \lambda_2 d(Nz, Kz)\frac{1 + d(Mz^*, Lz^*)}{1 + d(Nz, Mz^*)} \\ & + \lambda_3 d(Nz, Kz)\frac{1 + d(Nz, Lz^*) + d(Mz^*, Kz)}{1 + d(Nz, Kz) + d(Mz^*, Lz^*)} \\ & + \lambda_4 maz\left\{d(Nz, Mz^*), d(Nz, Kz), d(Mz^*, Lz^*)\right\}, \end{aligned}$$

$$\Rightarrow \quad d(z, z^*) \precsim \lambda_4 d(z, z^*),$$

which is contradiction, thus $z = z^*$ and hence z is a unique common fixed point of K, L, M and N. □

From Theorem 3.1, we can derived the following corollary by setting $K = L$ and $M = N$.

Corollary 3.1. *Let (X, d) be a complex valued metric space and $K, M : X \to X$ be two self-mappings satisfying the following conditions:*

I *the pair (K, M) satisfies (CLR_K) property;*

II

$$\begin{aligned} d(Kx, Ky) \precsim\ & \lambda_1 d(My, Ky)\frac{1 + d(Mx, Kx)}{1 + d(Mx, My)} + \lambda_2 d(Mx, Kx)\frac{1 + d(My, Ky)}{1 + d(Mx, My)} \\ & + \lambda_3 d(Mx, Kx)\frac{1 + d(Mx, Ky) + d(My, Kx)}{1 + d(Mx, Kx) + d(My, Ky)} \\ & + \lambda_4 max\left\{d(Mx, My), d(Mx, Kx), d(My, Ky)\right\}, \end{aligned}$$

where $x, y \in X$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, 1)$ such that $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$. If $K(X) \subseteq M(X)$, then the mapping K and M have common coincident point in X. Moreover if the pairs (K, M) is weakly compatible, then the mapping K and M have unique common fixed point in X.

Theorem 3.2. *Let (X, d) be a complex valued metric space and $K, L, N, M : X \to X$ be four self-mappings satisfying the following conditions*:

I *one of the pairs* (K, N) *and* (L, M) *satisfies property* $(E.A)$ *such that* $K(X) \subseteq M(X)$ *and* $L(X) \subseteq N(X)$;

II

$$d(Kx, Ly) \precsim \lambda_1 d(My, Ly)\frac{1 + d(Nx, Kx)}{1 + d(Nx, My)} + \lambda_2 d(Nx, Kx)\frac{1 + d(My, Ly)}{1 + d(Nx, My)}$$
$$+ \lambda_3 d(Nx, Kx)\frac{1 + d(Nx, Ly) + d(My, Kx)}{1 + d(Nx, Kx) + d(My, Ly)}$$
$$+ \lambda_4 max\Big\{d(Nx, My), d(Nx, Kx), d(My, Ly)\Big\},$$

where $x, y \in X$ *and* $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, 1)$ *such that* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$. *If one of* $M(X)$ *and* $N(X)$ *is closed subspace of* X*, then the mapping* K, L, M *and* N *have unique common fixed point in* X.

Proof. Since the property $(E.A)$ together with the closed-ness property of a suitable subspace gives rise closed range property, therefore the proof of the present theorem follows on the lines of the proof of Theorem 3.1. □

Remark 3.1. *If we put* $\lambda_2 = \lambda_3 = 0$ *in Theorem 3.2, we get Theorem 3.1 of* [**8**].

Remark 3.2. *If we put* $\lambda_2 = \lambda_3 = 0$ *and setting* $K = L$ *and* $M = N$ *in Theorem 3.2, we get Corollary 3.2 of* [**8**].

Remark 3.3. *If we put* $\lambda_2 = \lambda_3 = 0$ *in Theorem 3.1, we get Theorem 4.1 of* [**8**].

Remark 3.4. *If we put* $\lambda_2 = \lambda_3 = 0$ *in Corollary 3.1, we get Corollary 4.2 of* [**8**].

Theorem 3.3. *Let* (X, d) *be a complex valued metric space and* $A, B, S, T, P, Q : X \to X$ *be six self-mappings satisfying the following conditions:*

I *either both the pairs* (A, S) *and* (A, Q) *satisfies common* (CLR_A) *property or both the pairs* (B, T) *and* (B, P) *satisfies common* (CLR_B) *property;*

II $A(X) \subseteq T(X), A(X) \subseteq P(X), B(X) \subseteq S(X)$ *and* $B(X) \subseteq Q(X)$;

III *for each* $x, y \in X$ *and* $0 < k < 1$,

$$d(Ax, By) \precsim kd(Sx, Ty)d(Sx, Ax)d(Ty, By)d(Qx, Py).$$

If the pairs $(A, S), (B, T), (A, Q)$ *and* (B, P) *are weakly compatible, then* A, B, S, T, P *and* Q *have a unique common fixed point in* X.

Proof. Suppose that the pairs (B, T) and (B, P) satisfies common (CLR_B) property. Then there exist two sequences $\{x_n\}$ and $\{x_n^*\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Bx_n^* = \lim_{n\to\infty} Px_n^* = Bt \text{ for some } t \in X. \tag{3.13}$$

Since $B(X) \subseteq S(X)$ and $B(X) \subseteq Q(X)$, so that

$$Su_1 = Bt \text{ for some } u_1 \in X \text{ and } Qu_2 = Bt \text{ for some } u_2 \in X. \tag{3.14}$$

We show that $Au_1 = Su_1$. For this, using condition (III) with $x = u_1$ and $y = x_n$, we have

$$d(Au_1, Bx_n) \precsim kd(Su_1, Tx_n)d(Su_1, Au_1)d(Tx_n, Bx_n)d(Qu_1, Px_n).$$

Taking limit as $n \to \infty$ and using (3.13), (3.14), we get $d(Au_1, Bt) \precsim 0$ implies that $Au_1 = Bt$. Thus $Au_1 = Su_1 = Bt$. But $A(X) \subseteq T(X)$, so there exists $v_1 \in X$ such that $Au_1 = Tv_1$ and hence

$$Au_1 = Su_1 = Tv_1 = Bt. \tag{3.15}$$

Next, we claim that $Tv_1 = Bv_1$. To support our claim, putting $x = u_1$ and $y = v_1$ in condition (III), we have

$$d(Au_1, Bv_1) \precsim kd(Su_1, Tv_1)d(Su_1, Au_1)d(Tv_1, Bv_1)d(Qu_1, Pv_1).$$

With the help of (3.15), we get $d(Tv_1, Bv_1) \precsim 0$, which is contradiction. Thus $Tv_1 = Bv_1$ and from (3.15), we get

$$Au_1 = Su_1 = Tv_1 = Bv_1 = Bt. \tag{3.16}$$

Also, we assert that $Au_2 = Qu_2$. For this, using triangular inequality, we have

$$d(Au_2, Bt) \precsim d(Au_2, Bx_n^*) + d(Bx_n^*, Bt),$$

using condition (III) with setting $x = u_2$ and $y = x_n^*$, we have

$$d(Au_2, Bt) \precsim kd(Su_2, Tx_n^*)d(Su_2, Au_2)d(Tx_n^*, Bx_n^*)d(Qu_2, Px_n^*) + d(Bx_n^*, Bt).$$

Taking $n \to \infty$ and using (3.13), (3.14), we get $d(Au_2, Bt) \precsim 0$. Thus $Au_2 = Bt$ implies that $Au_2 = Qu_2 = Bt$. But $A(X) \subseteq P(X)$, so there exists $v_2 \in X$ such that $Au_2 = Pv_2$ and hence

$$Au_2 = Qu_2 = Pv_2 = Bt. \tag{3.17}$$

Next, we claim that $Pv_2 = Bv_2$. To support our claim, setting $x = u_2$ and $y = v_2$ in condition (III), we have

$$d(Pv_2, Bv_2) = d(Au_2, Bv_2) \precsim kd(Su_2, Tv_2)d(Su_2, Au_2)d(Tv_2, Bv_2)d(Qu_2, Pv_2),$$

with the help of (3.17), we get $d(Pv_2, Bv_2) \precsim 0$ which is possible only if $d(Pv_2, Bv_2) = 0$, that is $Pv_2 = Bv_2$. Hence equation (3.17) becomes

$$Au_2 = Qu_2 = Pv_2 = Bv_2 = Bt. \tag{3.18}$$

Therefore from (3.16) and (3.18), one can write

$$Au_1 = Su_1 = Tv_1 = Bv_1 = Au_2 = Qu_2 = Pv_2 = Bv_2 = Bt = z(say). \tag{3.19}$$

Now, we show that z is the common fixed point of A, B, S, T, P and Q. For this, using the weak compatibility of the pairs (A, S), (B, T), (A, Q), (B, P) and equation (3.23), we have

$$Au_1 = Su_1 \Rightarrow ASu_1 = SAu_1 \Rightarrow Az = Sz. \tag{3.20}$$

$$Tv_1 = Bv_1 \Rightarrow BTv_1 = TBv_1 \Rightarrow Bz = Tz. \tag{3.21}$$

$$Au_2 = Qu_2 \Rightarrow AQu_2 = QAu_2 \Rightarrow Az = Qz. \tag{3.22}$$

$$Pv_2 = Bv_2 \quad \Rightarrow BPv_2 = PBv_2 \quad \Rightarrow \quad Bz = Pz. \tag{3.23}$$

To show that $Az = z$, setting $x = z$ and $y = v_1$ in condition (III), we have

$$d(Az, Bv_1) \precsim kd(Sz, Tv_1)d(Sz, Az)d(Tv_1, Bv_1)d(Qz, Pv_1),$$

using (3.20), we get $d(Az, z) \precsim 0 \Rightarrow Az = z$. Hence from (3.20) and (3.22), we get

$$Az = Sz = Qz = z \tag{3.24}$$

Similarly, to show that $Bz = z$, putting $x = u_1$ and $y = z$ in condition (III) and using equations (3.21), (3.23), we get

$$Bz = Tz = Pz = z \tag{3.25}$$

Therefor from (3.24) and (3.25), one can write

$$Az = Bz = Sz = Tz = Pz = Qz = z \tag{3.26}$$

That is z is the common fixed point of A, B, S, T, P and Q.

Similar argument arises if we assume that the pairs (A, S) and (A, Q) satisfies common (CLR_A) property.

Uniqueness: Assume that $z^* \neq z$ be another common fixed point of A, B, S, T, P and Q. Then using condition (III) with $x = z$ and $y = z^*$

$$d(Az, Bz^*) \precsim kd(Sz, Tz^*)d(Sz, Az)d(Tz^*, Bz^*)d(Qz, Pz^*),$$

implies that $d(z, z^*) \precsim 0$ or $|d(z, z^*)| \leqslant 0$, which is contradiction. Hence z is unique common fixed point of A, B, S, T, P and Q. □

By taking $A = B$ in Theorem 3.3, we get the following corollary:

Corollary 3.2. *Let (X, d) be a complex valued metric space and $A, S, T, P, Q : X \to X$ be five self-mappings satisfying the following conditions*:

- I *either the pairs (A, S) and (A, Q) or the pairs (A, T) and (A, P) satisfies common (CLR_A) property;*
- II $A(X) \subseteq T(X), A(X) \subseteq P(X), A(X) \subseteq S(X)$ *and* $A(X) \subseteq Q(X)$;
- III *for each $x, y \in X$ and $0 < k < 1$,*

$$d(Ax, Ay) \precsim kd(Sx, Ty)d(Sx, Ax)d(Ty, Ay)d(Qx, Py).$$

If the pairs $(A, S), (A, T), (A, Q)$ and (A, P) are weakly compatible, then A, S, T, P and Q have a unique common fixed point in X.

By taking $P = T$ and $Q = S$ in Theorem 3.3, we get the following corollary:

Corollary 3.3. *Let (X, d) be a complex valued metric space and $A, B, S, T : X \to X$ be four self-mappings satisfying the following conditions:*

- I *either (A, S) satisfies (CLR_A) property or (B, T) satisfies (CLR_B) property;*
- II $A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$*;*
- III *for each $x, y \in X$ and $0 < k < 1$,*

$$d(Ax, By) \precsim k[d(Sx, Ty)]^2 d(Sx, Ax)d(Ty, By).$$

If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

By taking $A = B$, $T = S$ and $P = Q$ in Theorem 3.3, we get the following corollary:

Corollary 3.4. *Let (X, d) be a complex valued metric space and $A, T, P : X \to X$ be three self-mappings satisfying the following conditions:*

I *the pairs (A, T) and (A, P) satisfies common (CLR_A) property;*

II $A(X) \subseteq T(X)$ *and* $A(X) \subseteq P(X)$*;*

III *for each* $x, y \in X$ *and* $0 < k < 1$,

$$d(Ax, Ay) \precsim kd(Tx, Ty)d(Tx, Ax)d(Ty, Ay)d(Px, Py).$$

If the pairs (A, T) and (A, P) are weakly compatible, then A, T and P have a unique common fixed point in X.

By taking $A = B$ and $T = S = P = Q$ in Theorem 3.3, we get the following corollary:

Corollary 3.5. *Let (X, d) be a complex valued metric space and $A, T : X \to X$ be two self-mappings satisfying the following conditions*:

I *the pair (A, T) satisfies (CLR_A) property*;

II $A(X) \subseteq T(X)$;

III *for each* $x, y \in X$ *and* $0 < k < 1$,

$$d(Ax, Ay) \precsim k[d(Tx, Ty)]^2 d(Tx, Ax)d(Ty, Ay).$$

If the pair (A, T) is weakly compatible, then A and T have a unique common fixed point in X.

Theorem 3.4. *Let (X, d) be a complex valued metric space and $A, B, S, T, P, Q : X \to X$ be six self-mappings satisfying the following conditions*:

I *either the pairs (A, S) and (A, Q) satisfies common $(E.A)$ property or the pairs (B, T) and (B, P) satisfies common $(E.A)$ property*;

II $A(X) \subseteq T(X), A(X) \subseteq P(X), B(X) \subseteq S(X)$ *and* $B(X) \subseteq Q(X)$ *such that either both $T(X)$ and $P(X)$ are closed subspaces of X or both $S(X)$ and $Q(X)$ are closed subspaces of X*;

III *for each* $x, y \in X$ *and* $0 < k < 1$,

$$d(Ax, By) \precsim kd(Sx, Ty)d(Sx, Ax)d(Ty, By)d(Qx, Py).$$

If the pairs $(A, S), (B, T), (A, Q)$ and (B, P) are weakly compatible, then A, B, S, T, P and Q have a unique common fixed point in X.

Proof. Since the common $(E.A)$ property together with the closed-ness property of a suitable subspace gives rise closed range property, therefore the proof of the present theorem follows on the lines of the proof of Theorem 3.3. □

Remark 3.5. *Theorem 3.3 and Theorem 3.4 generalizes Theorem 10 of* [**10**].

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