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Directed edge - graceful labeling of cycle and star related graphs

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Abstract

Rosa [12] introduced the notion of graceful labelings. In 1985, Lo [11] introduced the notion of edge – graceful graphs. We extended the concept of edge – graceful labelings to directed graphs in [8]. In this paper we investigate directed edge – graceful labeling of cycle and star related graphs.

Key words: Graceful graphs, Directed graphs, Edge-graceful labeling
AMS Subject Classification (2010): 05C78

1 Introduction

All graphs in this paper are finite and directed. Terms not defined here are used in the sense of Harary [9]. The symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p, q) graph.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications [1, 2]. A good account on graceful labeling problems can be found in the dynamic survey of Gallian [6].

A graph G is called a graceful labeling if f is an injection from the vertices of G to the set $\{0, 1, 2, ..., q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

A graph $G(V, E)$ is said to be edge–graceful if there exists a bijection f from E to $\{1, 2, ..., |E|\}$ such that the induced mapping f^+ from V to $\{0, 1, ..., |V| - 1\}$ given by, $f^+(x) = (\Sigma f(xy)) \bmod(|V|)$ taken over all edges xy incident at x is a bijection.

A necessary condition for a graph G with p vertices and q edges to be edge–graceful is $q(q + 1) \equiv \frac{p(p+1)}{2} (\bmod\ p)$. Gayathri and Duraisamy introduced the concept of even edge-graceful labeling in [7]. Bloom and Hsu [3, 4, and 5] extended the notion of graceful labeling to directed graph. The concept of magic, antimagic and conservative labelings have been extended to directed graphs [10]. In [8] we extended the concept of edge–graceful labelings to directed graphs. In this paper we investigate directed edge – graceful labeling of cycle and star related graphs.

A (p, q) graph G is said to be directed edge – graceful if there exists an orientation of G and a labeling f of the arcs A of G with $\{1, 2, ..., q\}$ such that induced mapping g on V defined by, $g(v) = \left[f^+(v) - f^-(v) \right]$ (mod p) is a bijection where, $f^+(v)$ = the sum of the labels of all arcs with head v and $f^-(v)$ = the sum of the labels of all arcs with v as tail.

A graph G is said to be directed edge–graceful graph if it has directed edge–graceful labelings. Here, we investigate directed edge – graceful labeling of cycle and star related graphs.

2 Prior Results

Theorem 2.1. [8] *The path P_{2n+1} is directed edge-graceful for all $n \geq 1$.*

Theorem 2.2. [8] *The cycle graph C_{2n+1} is directed edge-graceful for all $n \geq 1$.*

Theorem 2.3. [8] *The Butterfly graph B_n is directed edge-graceful if n is odd.*

Theorem 2.4. [8] *The Butterfly graph B_n is directed edge – graceful if n is even and $n \geq 4$.*

Theorem 2.5. [8] *The snail graph $SN(2n + 1)$ is directed edge-graceful for all $n \geq 1$.*

Theorem 2.6. [8] $\left\langle K_{1,n} : K_{1,n} \right\rangle$ *is directed edge-graceful if n is even and $n \geq 4$.*

Theorem 2.7. [8] *The graph $P_3 \cup K_{1,2n+1}$ is directed edge-graceful for all $n \geq 1$.*

Theorem 2.8. [8] *The graph P_{2m} @ $K_{1,2n+1}$ is directed edge-graceful for all $m \geq 2$ and $n \geq 1$.*

Theorem 2.9. [8] The graph P_{2m+1} @ $K_{1,2n}$ is directed edge-graceful for all $m \geq 1$ and $n \geq 1$.

3 Main Results

Definition 3.1. *G_1 @ G_2 is nothing but one point union of G_1 and G_2.*

Theorem 3.2. *The graph C_{2m} @ $K_{1, 2n+1}$ is directed edge-graceful for $m \geq 2$ and $n \geq 1$.*

Proof. Let $G = C_{2m}$ @ $K_{1, 2n+1}$ and $V[C_{2m}$ @ $K_{1, 2n+1}] = \{v_1, v_2, ..., v_{2m}, u_1, u_2, ..., u_{2n+1}\}$ be the set of vertices. Now we orient the edges of C_{2m} @ $K_{1, 2n+1}$ such that the arc set A is given by,

$A = \{(v_{2i+1}, v_{2i}), 1 \leq i \leq m-1\} \cup \{v_1, v_{2m})\} \cup \{(v_{2i-1}, v_{2i}), 1 \leq i \leq m\} \cup \{(v_1, u_j), 1 \leq j \leq 2n+1\}$.

The edges and their orientation of C_{2m} @ $K_{1,2n+1}$ are as in Figure 1.

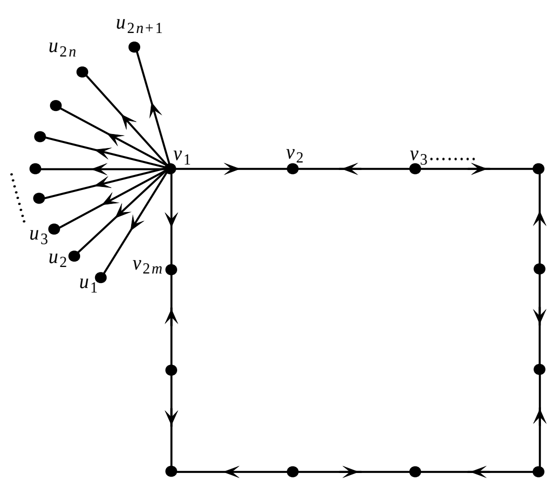

Figure 1: C_{2m} @ $K_{1,2n+1}$ with orientation

We now label the arcs of A as follows:

$f((v_{2i+1}, v_{2i})) = i, 1 \leq i \leq m - 1$

$f((v_1, v_{2m})) = m$

$f((v_{2i-1}, v_{2i})) = m + 2n + 1 + i, 1 \leq i \leq m$

$f((v_1, u_{2j-1})) = m + j, 1 \leq j \leq n + 1$

$f((v_1, u_{2j})) = m + 2n + 2 - j, \; 1 \leq j \leq n$

Then the values of $f^+(v_i)$, $f^+(u_j)$ and $f^-(v_i)$, $f^-(u_j)$ are computed as under.

$f^+(v_{2i}) = m + 2n + 1 + 2i, 1 \leq i \leq m$

$f^-(v_{2i}) = 0, 1 \leq i \leq m$

$f^+(v_{2i+1}) = 0, 1 \leq i \leq m - 1$

$f^-(v_{2i+1}) = -(m + 2n + 2 + 2i), 1 \leq i \leq m - 1$

$f^+(v_1) = 0$

$f^-(v_1) = -(m + n + 1)\,[2(n + 1) + 1]$

$f^+(u_{2j-1}) = m + j, 1 \leq j \leq n + 1$

$f^-(u_{2j-1}) = 0, 1 \leq j \leq n + 1$

$f^+(u_{2j}) = m + 2n + 2 - j, 1 \leq j \leq n$

$f^-(u_{2j}) = 0, 1 \leq j \leq n$

Then the induced vertex labels are,

$g(u_{2j-1}) = m + j, 1 \leq j \leq n + 1$

$g(u_{2j}) = m + 2n + 2 - j, 1 \leq j \leq n$

Case (i) m is odd.

$$g(v_{2i-1}) = m + 1 - 2i, 1 \leq i \leq \frac{m+1}{2}$$

$g(v_{2i})$ $= m + 2n + 1 + 2i,\ 1 \le i \le \dfrac{m-1}{2}$

$g(v_{m-1+2i})$ $= 2i - 1,\ 1 \le i \le \dfrac{m+1}{2}$

$g(v_{m+2i})$ $= (2m + 2n + 1) - \left(\dfrac{m-1}{2}\right) + 2 - 2i,\ 1 \le i \le \dfrac{m-1}{2}$

Case (ii) *m* is even.

$g(v_{2i-1})$ $= m + 1 - 2i,\ 1 \le i \le \dfrac{m}{2}$

$g(v_{2i})$ $= m + 2n + 1 + 2i,\ 1 \le i \le \dfrac{m-2}{2}$

$g(v_{m-2+2i})$ $= 2i - 2,\ 1 \le i \le \dfrac{m+2}{2}$

$g(v_{m-1+2i})$ $= 2m + 2n + 2 - 2i,\ 1 \le i \le \dfrac{m}{2}$.

Clearly, $g(V)$ $= \{0, 1, ..., 2m + 2n\} = \{0, 1, ..., p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, C_{2m} @ $K_{1,2n+1}$ is a directed edge - graceful graph. The directed edge - graceful labeling of C_{10} @ $K_{1,13}$ and C_{12} @ $K_{1,9}$ are given in Figure 2 and Figure 3 respectively.

■

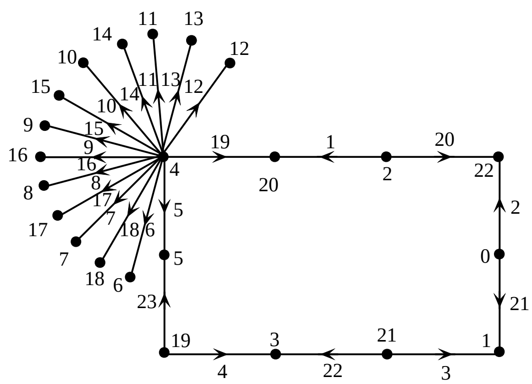

Figure 2: C_{10} @ $K_{1,13}$ with directed edge - graceful labeling

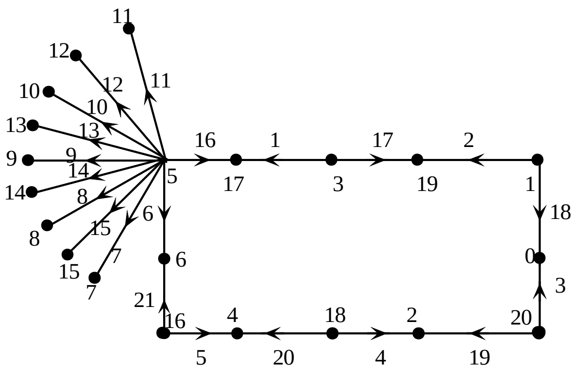

Figure 3: C_{12} @ $K_{1,9}$ with directed edge - graceful labeling

Theorem 3.3. *The graph* C_{2m+1} @ $K_{1,2n}$ *is a directed edge - graceful for* $m \geq 1$ *and* $n \geq 1$.

Proof. Let $G = C_{2m+1}$ @ $K_{1,2n}$ and $V[C_{2m+1}$ @ $K_{1,2n}] = \{v_1, v_2, ..., v_{2m+1}, u_1, u_2, ..., u_{2n}\}$ be the set of vertices. Now we orient the edges of C_{2m+1} @ $K_{1,2n}$ such that the arc set A is given by,

$A = \{(v_{2i-1}, v_{2i}), 1 \leq i \leq m\} \cup \{(v_{2i+1}, v_{2i}), 1 \leq i \leq m\} \cup \{(v_{2m+1}, v_1)\} \cup \{(v_{m+1}, u_j), 1 \leq j \leq 2n\}$

The edges and their orientation of C_{2m+1} @ $K_{1,2n}$ are as in Figure 4.

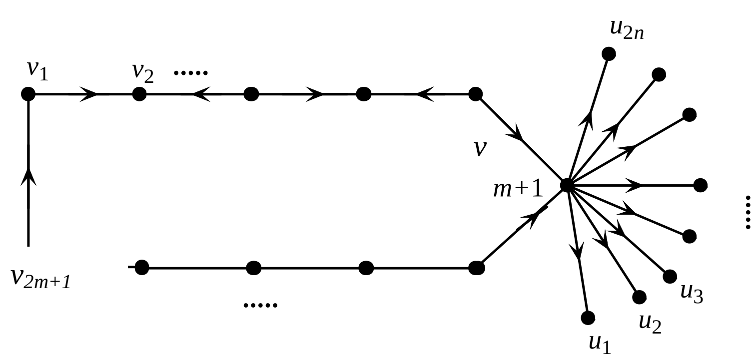

Figure 4: C_{2m+1} @ $K_{1,2n}$ with orientation

We now label the arcs of A as follows:

$f((v_{2i-1}, v_{2i})) = m + 2n + i, 1 \le i \le m$

$f((v_{2i+1}, v_{2i})) = i, 1 \le i \le m$

$f((v_{2m+1}, v_1)) = 2m + 2n + 1$

$f((v_{m+1}, u_1)) = m + 1$

$f((v_{m+1}, u_{2j})) = m + 1 + j, 1 \le j \le n$

$f((v_{m+1}, u_{2j+1}))= m + 2n + 1 - j, 1 \le j \le n - 1$

Then the values of $f^+(v_i)$, $f^+(u_j)$ and $f^-(v_i)$, $f^-(u_j)$ are computed as under.

$$f^+(v_1) = 2m + 2n + 1$$

$$f^-(v_1) = -(m + 2n + 1)$$

$$f^+(v_{2i}) = \begin{cases} m + 2n + 2i, 1 \le i \le \frac{m}{2} \text{ if } m \text{ is even} \\ m + 2n + 2i, 1 \le i \le \frac{m-1}{2} \text{ if } m \text{ is odd} \end{cases}$$

$$f^-(v_{2i}) = \begin{cases} 0, 1 \le i \le \frac{m}{2} \text{ if } m \text{ is even.} \\ 0, 1 \le i \le \frac{m-1}{2} \text{ if } m \text{ is odd} \end{cases}$$

$$f^+(v_{m+1}) = \begin{cases} 2m + 2n + 1, \text{ if m is odd} \\ 0, \text{ if m is even} \end{cases}$$

$$f^-(v_{m+1}) = \begin{cases} -[(n - 1)(2m + 2n + 2) + 2(m + 1) + n], \text{ if } m \text{ is odd} \\ -[n(2m + 2n + 3) + 2m + 1], \text{ if } m \text{ is even} \end{cases}$$

$$f^+(v_{m+2i}) = \begin{cases} 2m + 2n + 2i, 1 \le i \le \frac{m}{2} \text{ if } m \text{ is even} \\ 0, 1 \le i \le \frac{m+1}{2} \text{ if } m \text{ is odd} \end{cases}$$

$$f^-(v_{m+2i}) = \begin{cases} 0,\ 1 \le i \le \frac{m}{2} & \text{if } m \text{ is even} \\ -(2m + 2n + 2i),\ 1 \le i \le \frac{m+1}{2} & \text{if } m \text{ is odd} \end{cases}$$

$$f^+(v_{m+1+2i}) = \begin{cases} 0,\ 1 \le i \le \frac{m}{2} & \text{if } m \text{ is even} \\ 2m + 2n + 1 + 2i,\ 1 \le i \le \frac{m-1}{2} & \text{if } m \text{ is odd} \end{cases}$$

$$f^-(v_{m+1+2i}) = \begin{cases} -(2m + 2n + 1 + 2i),\ 1 \le i \le \frac{m}{2} & \text{if } m \text{ is even} \\ 0,\ 1 \le i \le \frac{m-1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

$f^+(u_1) = m + 1$

$f^-(u_1) = 0$

$f^+(u_{2j}) = m + 1 + j,\ 1 \le j \le n$

$f^-(u_{2j}) = 0,\ 1 \le j \le n$

$f^+(u_{2j+1}) = m + 2n + 1 - j,\ 1 \le j \le n - 1$

$f^-(u_{2j+1}) = 0,\ 1 \le j \le n - 1$

Then the induced vertex labels are

$g(u_1) = m + 1$

$g(u_{2j}) = m + 1 + j,\ 1 \le j \le n$

$g(u_{2j+1}) = m + 2n + 1 - j,\ 1 \le j \le n - 1$

Case (i) *m* is odd

$g(v_{2i-1}) = m + 2 - 2i,\ 1 \le i \le \frac{m+1}{2}$

$g(v_{2i}) = m + 2n + 2i,\ 1 \le i \le \frac{m-1}{2}$

$g(v_{m+1}) = 0$

$g(v_{m+2i}) \quad = 2m + 2n + 2 - 2i,\ 1 \le i \le \frac{m+1}{2}$

$g(v_{m+1+2i}) \quad = 2i,\ 1 \le i \le \frac{m-1}{2}$

Case (ii) *m* is even

$g(v_{2i-1}) \quad = m + 2 - 2i,\ 1 \le i \le \frac{m}{2}$

$g(v_{2i}) \quad = m + 2n + 2i,\ 1 \le i \le \frac{m}{2}$

$g(v_{m+1}) \quad = 0$

$g(v_{m+2i}) \quad = 2i - 1,\ 1 \le i \le \frac{m}{2}$

$g(v_{m+1+2i}) \quad = 2m + 2n + 1 - 2i,\ 1 \le i \le \frac{m}{2}$

Clearly, $g(V) = \{0, 1, ..., 2m + 2n\} = \{0, 1, ..., p - 1\}$

So, it follows that all the vertex labels are distinct and g is a bijection. Hence, $C_{2m+1} @ K_{1,2n}$ is a directed edge-graceful graph. ■

The directed edge-graceful labeling of $C_9 @ K_{1,10}$ and $C_{11} @ K_{1,8}$ are given in Figure 5 and Figure 6 respectively.

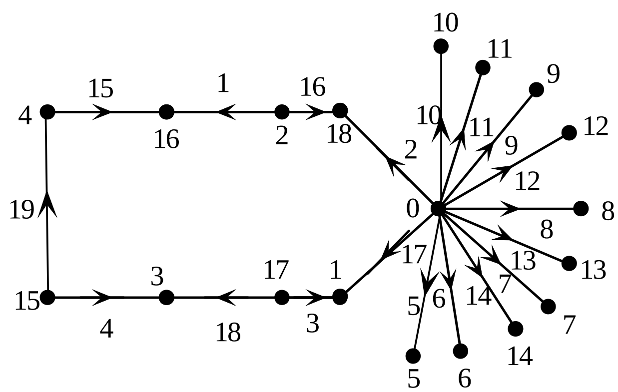

Figure 5: $C_9 @ K_{1,10}$ with directed edge - graceful labeling

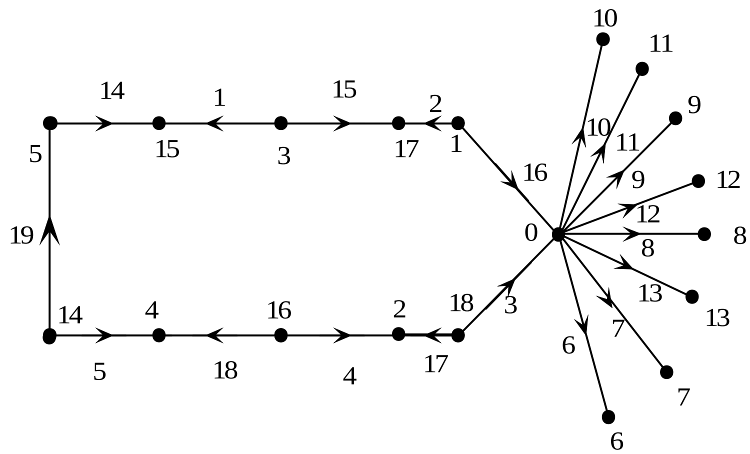

Figure 6: C_{11} @ $K_{1,8}$ with directed edge - graceful labeling

Definition 3.4. *If G has order n, the corona of G with H denoted by G⊙H is the graph obtained by taking one copy of G and n copies of H and joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H.*

Theorem 3.5. *The graph $C_n \odot \bar{K}_m$ is directed edge-graceful if n is odd and m is even for* $n \geq 3$ & $m \geq 2$.

Proof. Let $G = C_n \odot \bar{K}_m$ and $V[C_n \odot \bar{K}_m] = \{v_1, v_2, ..., v_n, v_{11}, v_{12}, ..., v_{1m}, v_{21}, v_{22}, ..., v_{2m}, ..., v_{n1}, v_{n2}, ..., v_{nm}\}$ be the set of vertices. Now we orient the edges of $C_n \odot \bar{K}_m$ such that the arc set A is given by,

$$A = \left\{(v_{2i-1}, v_{2i}),\ 1 \leq i \leq \frac{n-1}{2}\right\} \cup \left\{(v_{2i+1}, v_{2i}),\ 1 \leq i \leq \frac{n-1}{2}\right\} \cup \{(v_n, v_1)\} \cup$$

$$\{(v_i, v_{i,j}),\ 1 \leq i \leq n,\ 1 \leq j \leq m\}$$

The edges and their orientation of $C_n \odot \bar{K}_m$ are as in Figure 7.

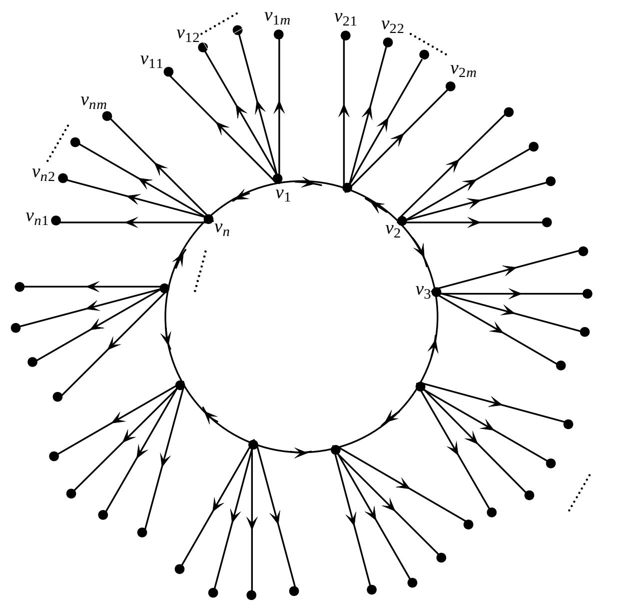

Figure 7: $C_n \odot \bar{K}_m$ with orientation

We now label the arcs of A as follows:

$$f(v_{2i+1}, v_{2i}) = i,\ 1 \le i \le \frac{n-1}{2}$$

$$f(v_{2i-1}, v_{2i}) = n(m+1) - \left(\frac{n-1}{2}\right) - 1 + i,\ 1 \le i \le \frac{n-1}{2}$$

$$f(v_n, v_1) = n(m+1)$$

$$f(v_i, v_{i(2j-1)}) = \left(\frac{n-1}{2}\right) + \frac{m}{2}(i-1) + j,\ 1 \le i \le n,\ 1 \le j \le \frac{m}{2}$$

$$f(v_i, v_{i(2j)}) = \left[n(m+1) - \left(\frac{n-1}{2}\right)\right] - \frac{m}{2}(i-1) - j,\ 1 \le i \le n, 1 \le j \le \frac{m}{2}$$

Then the values of $f^+(v_i)$, $f^+(v_{ij})$ and $f^-(v_i)$, $f^-(v_{i,j})$ are computed as under

$$f^+(v_1) = n(m+1)$$

$$f^-(v_1) = -\left[(nm+n)\left(\frac{m}{2}+1\right)-\left(\frac{n-1}{2}\right)\right]$$

$$f^+(v_{2i}) = (nm+n)-\left(\frac{n-1}{2}\right)-1+2i,\ 1\le i\le \frac{n-1}{2}$$

$$f^-(v_{2i}) = -\left[\frac{m}{2}(nm+n)\right],\ 1\le i\le \frac{n-1}{2}$$

$$f^+(v_{2i+1}) = 0,\ 1\le \text{i} \le \frac{n-1}{2}$$

$$f^-(v_{2i+1}) = -\left[(nm+n)\left(\frac{m}{2}+1\right)-\left(\frac{n-1}{2}\right)+2i\right],\ 1\le i\le \frac{n-1}{2}$$

$$f^+(v_{i(2j-1)}) = \left(\frac{n-1}{2}\right)+\frac{m}{2}(i-1)+j,\ 1\le i\le n,\ 1\le j\le \frac{m}{2}$$

$$f^-(v_{i(2j-1)}) = 0,\ 1\le i\le n,\ 1\le j\le \frac{m}{2}$$

$$f^+(v_{i(2j)}) = \left[n(m+1)-\left(\frac{n-1}{2}\right)\right]-\frac{m}{2}(i-1)-j, 1\le i\le n, 1\le j\le \frac{m}{2}$$

$$f^-(v_{i(2j)}) = 0,\ 1\le i\le n,\ 1\le j\le \frac{m}{2}$$

Then the induced vertex labels are,

$$g(v_{i(2j-1)}) = \left(\frac{n-1}{2}\right)+\frac{m}{2}(i-1)+j,\ 1\le i\le n,\ 1\le j\le \frac{m}{2}$$

$$g(v_{i(2j)}) = \left[n(m+1)-\left(\frac{n-1}{2}\right)\right]-\frac{m}{2}(i-1)-j, 1\le i\le n, 1\le j\le \frac{m}{2}$$

Case (i) $\left(\frac{n-1}{2}\right)$ is even.

$$g(v_{2i-1}) = \left(\frac{n-1}{2}\right) + 2 - 2i,\ 1 \le i \le \frac{n+3}{4}$$

$$g(v_{2i}) = n(m+1) - \left(\frac{n-1}{2}\right) - 1 + 2i,\ 1 \le i \le \frac{n-1}{4}$$

$$g\left(v_{\frac{n-1}{2}+2i}\right) = 2i - 1,\ 1 \le i \le \frac{n-1}{4}$$

$$g\left(v_{\frac{n+1}{2}+2i}\right) = n(m+1) - 2i,\ 1 \le i \le \frac{n-1}{4}$$

Case (ii) $\left(\frac{n-1}{2}\right)$ is odd.

$$g(v_{2i-1}) = \left(\frac{n-1}{2}\right) + 2 - 2i,\ 1 \le i \le \frac{n+1}{4}$$

$$g(v_{2i}) = n(m+1) - \left(\frac{n-1}{2}\right) - 1 + 2i,\ 1 \le i \le \frac{n-3}{4}$$

$$g\left(v_{\left(\frac{n-3}{2}\right)+2i}\right) = 2i - 2,\ 1 \le i \le \frac{n+1}{4}$$

$$g\left(v_{\left(\frac{n-1}{2}\right)+2i}\right) = n(m+1) + 1 - 2i,\ 1 \le i \le \frac{n+1}{4}$$

Clearly, $g(V) = \{0, 1, ..., nm + n - 1\} = \{0, 1, ..., p - 1\}$.

So, it follows that all the vertex labels are distinct and g is bijection. Hence, $C_n \odot \bar{K}_m$ is a directed edge-graceful graph. ■

The directed edge-graceful labeling of $C_5 \odot \bar{K}_8$ and $C_7 \odot \bar{K}_8$ are given in Figure 8 and Figure 9 respectively.

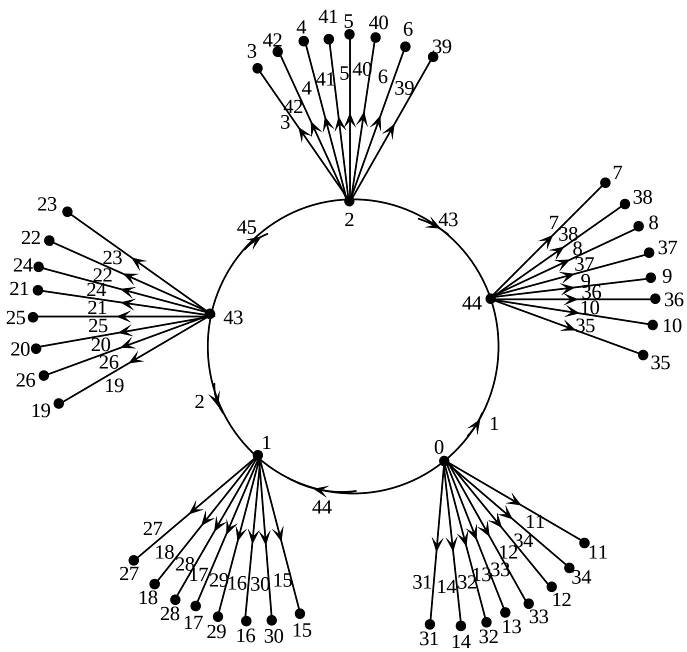

Figure 8: $C_5 \odot \overline{K}_8$ with directed edge - graceful labeling

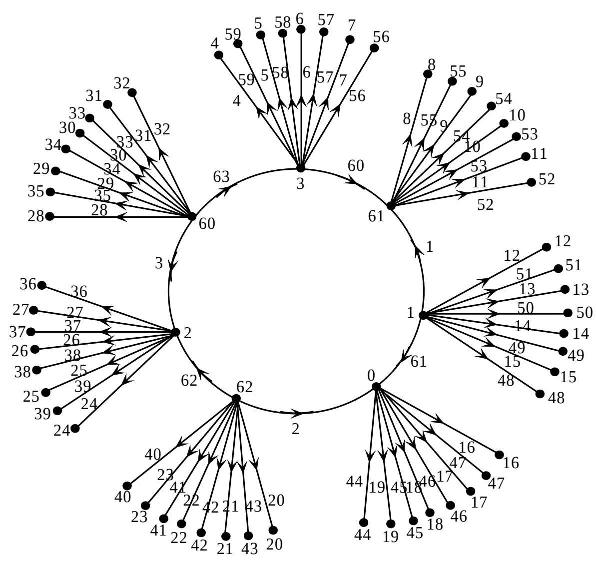

Figure 9: $C_7 \odot \overline{K}_8$ with directed edge - graceful labeling

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