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Cordial labeling for the splitting graph of some standard graphs

P. Lawrence Rozario Raj
Department of Mathematics, St. Joseph's College
Trichirappalli – 620 002, Tamil Nadu, India.
E-mail: lawraj2006@yahoo.co.in

S. Koilraj
Department of Mathematics, St. Joseph's College
Trichirappalli – 620 002, Tamil Nadu, India.
E-mail: skoilraj@yahoo.com

Abstract

In this paper we prove that the splitting graph of path P_n, cycle C_n, complete bipartite graph $K_{m,n}$, matching M_n, wheel W_n and $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : ... : K_{1,n}^{(k)} >$ are cordial.

Key words: Cordial labeling, Splitting graph.

AMS Subject Classification (2010): 05C78.

1 Introduction

All graphs considered here are finite, simple and undirected. The origin of graph labelings can be attributed to Rosa [7]. For all terminologies and notations we follow Harary [5]. Following definitions are useful for the present study.

Definition 1.1. [8] *For each vertex v of a graph G, take a new vertex v′. Join v′ to all the vertices of G adjacent to v. The graph S(G) thus obtained is called splitting graph of G.*

Definition 1.2. [9] *The graph* $G = < K_{1,n}^{(1)}:K_{1,n}^{(2)}:...:K_{1,n}^{(k)} >$ *is obtained from k copies of stars* $K_{1,n}^{(1)},K_{1,n}^{(2)},...,K_{1,n}^{(k)}$ *by joining apex vertices of each* $K_{1,n}^{(p-1)}$ *and* $K_{1,n}^{(p)}$ *to a new vertex* x_{p-1}, $2 \le p \le k$.

Note that G has $k(n + 2) - 1$ vertices and $k(n + 2) - 2$ edges.

Definition 1.3. *The assignment of values subject to certain conditions to the vertices of a graph is known as graph labeling.*

Definition 1.4. *Let G = (V, E) be a graph. A mapping f : V(G) →{0,1} is called binary vertex labeling of G and f(v) is called the label of the vertex v of G under f.*

For an edge e = uv, the induced edge labeling $f^*: E(G) \rightarrow \{0,1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.5. *A binary vertex labeling of a graph G is called a cordial labeling if* $| v_f(0) - v_f(1) | \leq 1$ *and* $| e_f(0) - e_f(1) | \leq 1$. *A graph G is cordial if it admits cordial labeling.*

Definition 1.6. *A wheel graph* W_n *is obtained from a cycle* C_n *by adding a new vertex and joining it to all the vertices of the cycle by an edge, the new edges are called the spokes of the wheel.*

Definition 1.7. *A fan graph* F_n *is obtained from a path* P_n *by adding a new vertex and joining it to all the vertices of the path by an edge, the new edges are called the spokes of the fan.*

Definition 1.8. *A matching graph* M_n *is n copies of* K_2.

The concept of cordial labeling was introduced by Cahit [3]. S.M. Lee and A. Liu [6] proved that all complete bipartite graphs and all fans are cordial. Further, they proved that, the cycle C_n is cordial if and only if $n \not\equiv 2 \pmod 4$, the matching M_n is cordial if and only if $n \not\equiv 2 \pmod 4$ and the wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$, $n \geq 3$. S.K. Vaidya et al.[9] proved $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$ is cordial.

In this paper, we prove that the splitting graph of path P_n, cycle C_n, complete bipartite graph $K_{m,n}$, matching M_n, wheel W_n and $< K_{1,n}^{(1)} : K_{1,n}^{(2)} : \ldots : K_{1,n}^{(k)} >$ are cordial.

2 Main Results

Theorem 2.1. *The graph $S(P_n)$ is cordial*.

Proof. Let G be P_n. The vertices of P_n are $v_1,v_2,\ldots,v_n$. Then $S(G)$ has the vertices $v_1,v_2,\ldots,v_n,v_1',v_2',\ldots,v_n'$. The vertex labeling $f : V(S(G)) \rightarrow \{0,1\}$ is given below.

$$f(v_i) = \begin{cases} 1 & \text{if } i \equiv 0, 1 \pmod 4 \\ 0 & \text{if } i \equiv 2, 3 \pmod 4 \end{cases}$$

$$f(v_i') = \begin{cases} 1 & \text{if } i \equiv 2, 3 \pmod 4 \\ 0 & \text{if } i \equiv 0, 1 \pmod 4 \end{cases}$$

$v_f(0) = v_f(1)$ for all n and $e_f(0) = e_f(1) + 1$ if n is even and

$e_f(0) = e_f(1)$ if n is odd.

Therefore the graph $S(G)$ satisfies the conditions $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$.

Hence $S(P_n)$ is cordial. ■

Illustration 2.2. The cordial labelings of $S(P_4)$ and $S(P_5)$ are shown in Figure 1(a) and 1(b).

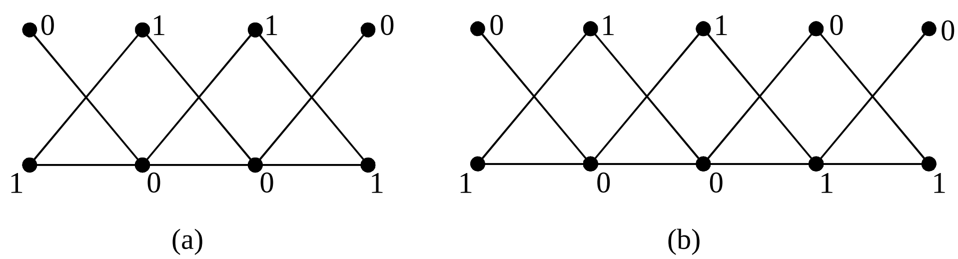


Figure 1: Cordial labelings of $S(P_4)$ and $S(P_5)$

Theorem 2.3. *The graph $S(C_n)$ is cordial for $n \not\equiv 2 \pmod 4$, $n \geq 3$.*

Proof. Let G be C_n ($n \geq 3$). The vertices of C_n are $v_1,v_2,\ldots,v_n$. Then $S(G)$ has the vertices $v_1,v_2,\ldots,v_n$, $v_1',v_2',\ldots,v_n'$. The vertex labeling f: $V(S(G)) \rightarrow \{0,1\}$ is given below.

$f(v_i) = 0$ and $f(v_i') = 1$ if $i \equiv 2, 3 \pmod 4$,

$f(v_i) = 1$ and $f(v_i') = 0$ if $i \equiv 0, 1 \pmod 4$.

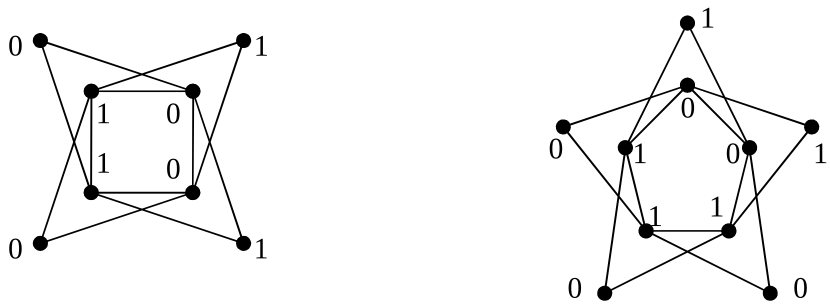
The following table shows that the graph $S(G)$ satisfies the conditions

$|v_f(0) - v_f(1)| \leq 1$ and $| e_f(0) - e_f(1)| \leq 1$.

n	Vertex Conditions	Edge Conditions
$n \equiv 0 \pmod 4$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
n is odd	$v_f(0) = v_f(1)$	$e_f(1) = e_f(0) + 1$

Hence $S(C_n)$ is cordial. ■

Illustration 2.4. The cordial labelings of $S(C_4)$ and $S(C_5)$ are shown in Figure 2 (a) and 2(b).



(a) (b)

Figure 2: Cordial labelings of $S(C_4)$ and $S(C_5)$

Theorem 2.5. *The graph $S(W_n)$ is cordial for $n \not\equiv 2 \pmod 4$, $n \geq 3$.*

Proof. Let G be W_n ($n \geq 3$). The vertices are $c, v_1, v_2, \ldots, v_n$. Then $S(G)$ has the vertices $c, v_1, v_2, \ldots, v_n, c', v_1', v_2', \ldots, v_n'$. The vertex labeling $f : V(S(G)) \rightarrow \{0,1\}$ is given below.

$f(c) = 0$ and $f(c') = 1$

Case (i) $n \equiv 0 \pmod 4$

$f(v_i) = f(v_i') = 1$ if $i \equiv 1, 2 \pmod 4$,

$f(v_i) = f(v_i') = 0$ if $i \equiv 0, 3 \pmod 4$.

Case (ii) $n \equiv 1 \pmod 4$

$f(v_i) = 0$ if $i \equiv 2, 3 \pmod 4$,

$f(v_i) = 1$ if $i \equiv 0, 1 \pmod 4$,

$f(v_i') = 1$ for $i = 1$ to $(n-1)/2$,

$f(v_i') = 0$ for $i = (n+1)/2$ to n.

Case (iii) $n \equiv 3 \pmod 4$

$f(v_i) = 0$ if $i \equiv 2, 3 \pmod 4$,

$f(v_i) = 1$ if $i \equiv 0, 1 \pmod 4$,

$f(v_i') = 0$ for $i = 1$ to $(n-1)/2$,

$f(v_i') = 1$ for $i = (n+1)/2$ to n.

The following table shows that the graph $S(G)$ satisfies the conditions $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)|\leq 1$.

n	Vertex Conditions	Edge Conditions
$n \equiv 0 \pmod 4$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$
$n \equiv 1 \pmod 4$	$v_f(0) = v_f(1)$	$e_f(1) = e_f(0)$
$n \equiv 3 \pmod 4$	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$

Hence $S(W_n)$ is cordial. ■

Illustration 2.6. The cordial labelings of $S(W_4)$ and $S(W_5)$ are shown in Figure 3 (a) and 3(b).

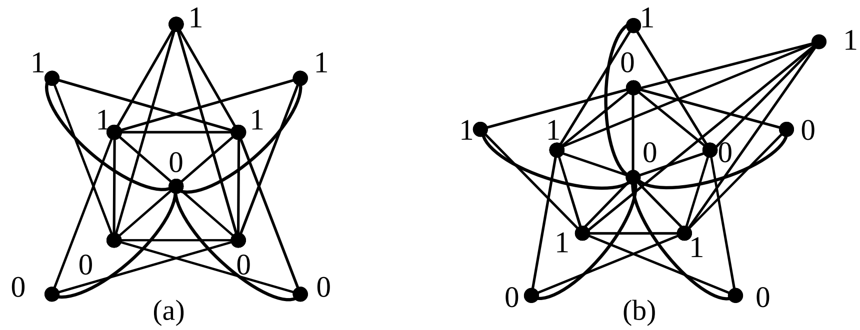


Figure 3: Cordial labelings of $S(W_4)$ and $S(W_5)$

Theorem 2.7. *The graph $S(M_n)$ is cordial.*

Proof. Let G be M_n. The vertices are $v_1, v_2, \ldots, v_{2n}$. Then $S(G)$ has the vertices $v_1, v_2, \ldots, v_{2n}, v_1', v_2', \ldots, v_{2n}'$ in the order v_2', v_1, v_2, v_1', v_4', v_3, v_4, v_3', …, v_{2n}', v_{2n-1}, v_{2n}, v_{2n-1}'. The vertex labeling $f : V(S(G)) \to \{0,1\}$ is given below.

$f(v_i) = 0$ if $i \equiv 0, 1, 2 \pmod 4$,

$f(v_i) = 1$ if $i \equiv 3 \pmod 4$,

$f(v_i') = 1$ if $i \equiv 0, 1, 2 \pmod 4$,

$f(v_i') = 0$ if $i \equiv 3 \pmod 4$.

The following table shows that the graph $S(G)$ satisfies the conditions $|v_f(0)–v_f(1)| \le 1$ and $|e_f(0)–e_f(1)|\le 1$.

n	Vertex Conditions	Edge Conditions
n is odd	$v_f(0) = v_f(1)$	$e_f(1) = e_f(0) + 1$
n is even	$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$

Hence S(M_n) is cordial. ■

Illustration 2.8. The cordial labelings of $S(M_3)$ and $S(M_4)$ are shown in Figure 4(a) and 4(b).

Figure 4: Cordial labelings of $S(M_3)$ and $S(M_4)$

Theorem 2.9. *The graph $S(F_n)$ is cordial for $n \ge 2$.*

Proof. Let G be F_n ($n \ge 2$). The vertices are $c,v_1,v_2,\ldots,v_n$. Then $S(G)$ has the vertices $c,v_1,v_2,\ldots,v_n, c',v_1',v_2',\ldots,v_n'$. The vertex labeling $f : V(S(G)) \to \{0,1\}$ is given below.

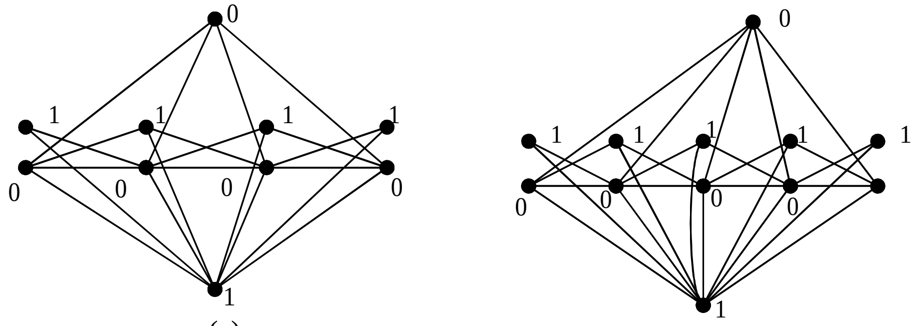
$f(c) = 1$ and $f(c') = 0$

$f(v_i) = 0$ and $f(v_i') = 1$ for $i = 1$ to n.

The graph $S(G)$ satisfies the conditions $|v_f(0)–v_f(1)|\le 1$ and $|e_f(0)–e_f(1)| \le 1$ since $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1) + 1$ for $n \ge 2$.

Hence $S(F_n)$ is cordial for $n \ge 2$. ■

Illustration 2.10. The cordial labelings of $S(F_4)$ and $S(F_5)$ are shown in Figure 5(a) and 5(b).



(a) (b)

Figure 5: Cordial labelings of $S(F_4)$ and $S(F_5)$

Theorem 2.11. *The graph $S(K_{m,n})$ is cordial for any m, $n \in N$.*

Proof. Let G be $K_{m,n}$. Denote the vertices of $K_{m,n}$ as $v_1,v_2,\ldots,v_m$ and $u_1,u_2,\ldots,u_n$. Then $S(G)$ has the vertices $v_1,v_2,\ldots,v_m,v_1',v_2',\ldots,v_m'$, $u_1,u_2,\ldots,u_n,u_1',u_2',\ldots,u_n'$. The vertex labeling $f : V(S(G)) \rightarrow \{0,1\}$ is given below.

$f(v_i) = f(u_i) = 1$ and $f(v_i') = f(u_i') = 0$ if i is odd.

$f(v_i) = f(u_i) = 0$ and $f(v_i') = f(u_i') = 1$ if i is even.

The following table shows that the graph S(G) satisfies the conditions $|v_f(0)–v_f(1)| \leq 1$ and $|e_f(0)–e_f(1)| \leq 1$.

m	n	Vertex Conditions	Edge Conditions
1	odd	$v_f(0) = v_f(1)$	$e_f(1) = e_f(0) + 1$
odd	1	$v_f(0) = v_f(1)$	$e_f(1) = e_f(0) + 1$
$m = n$ and odd		$v_f(0) = v_f(1)$	$e_f(1) = e_f(0) + 1$
others		$v_f(0) = v_f(1)$	$e_f(0) = e_f(1)$

Hence $S(K_{m,n})$ is cordial. ■

Illustration 2.12. The cordial labelings of $S(K_{2,3})$ and $S(K_{3,3})$ are shown in Figure 6(a) and 6(b).

(a) (b)

Figure 6: Cordial labelings of $S(K_{2,3})$ and $S(K_{3,3})$

Theorem 2.13. *The graph* $S(< K_{1,n}^{(1)}:K_{1,n}^{(2)}:...:K_{1,n}^{(k)} >)$ *is cordial*.

Proof. Let G be $< K_{1,n}^{(1)}:K_{1,n}^{(2)}:...:K_{1,n}^{(k)} >$. Let $K_{1,n}^{(i)}$, i = 1,2,..,k be copies of $K_{1,n}$. Let v_{ij} be the pendant vertices of $K_{1,n}^{(i)}$ and c_i be the apex vertex of $K_{1,n}^{(i)}$ (i = 1,2,..,k and j = 1,2,.., n) and $x_1, x_2,\ldots, x_{n-1}$ be vertices such that c_{i-1} and c_i are adjacent to x_{i-1}, where $2 \le i \le k$.

Now $S(G)$ has the vertices v_{ij}, v_{ij}', c_i, c_i', x_{i-1} and x_{i-1}' vertices, where i = 1,2,..,k and j = 1,2,..,n. The vertex labeling $f : V(S(G)) \to \{0,1\}$ is given below.

For i = 1,2,..,k

$f(v_{ij}) = 1$ and $f(v_{ij}') = 0$ if j is odd,

$f(v_{ij}) = 0$ and $f(v_{ij}') = 1$ if j is even,

$f(c_i) = 1$ and $f(c_i') = 0$ if i is odd,

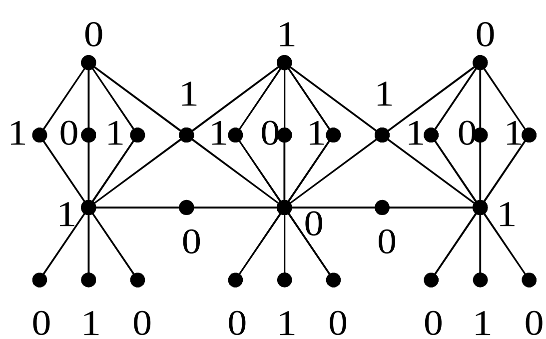
$f(c_i) = 0$ and $f(c_i') = 1$ if i is even,

$f(x_i) = 1$ and $f(x_i') = 0$ for i = 1 to n.

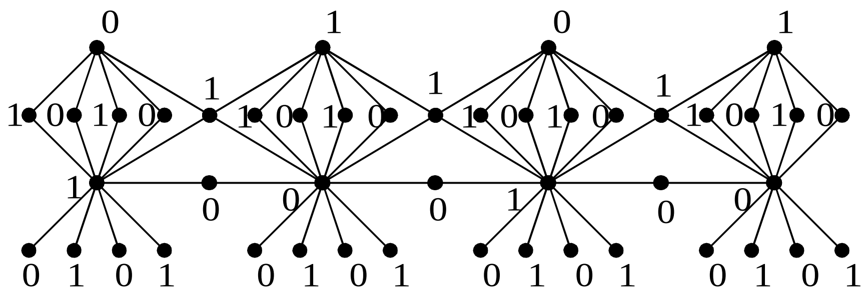
The graph $S(G)$ satisfies the conditions $| v_f(0) - v_f(1) | \le 1$ and $| e_f(0) - e_f(1) | \le 1$ since $v_f(0) = v_f(1)$ for all n and k and $e_f(1) = e_f(0) +1$ if n and k are odd, others $e_f(0) = e_f(1)$.

Hence $S(< K_{1,n}^{(1)}:K_{1,n}^{(2)}:...:K_{1,n}^{(k)} >)$ is cordial. ■

Illustration 2.14. The cordial labelings of $S(< K_{1,3}^{(1)}: K_{1,3}^{(2)}:K_{1,3}^{(3)} >)$ and $S(< K_{1,4}^{(1)}: K_{1,4}^{(2)}:K_{1,4}^{(3)}:K_{1,4}^{(4)} >)$ are shown in Figure 7(a) and 7(b).



(a)



(b)

Figure 7: Cordial labelings of $S(<K_{1,3}^{(1)}: K_{1,3}^{(2)}:K_{1,3}^{(3)}>)$ and

$S(<K_{1,4}^{(1)}: K_{1,4}^{(2)}:K_{1,4}^{(3)}:K_{1,4}^{(4)}>)$

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