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Equality of strong domination and chromatic strong domination in graphs

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Abstract

A subset D of a vertex set V of a graph G is a chromatic strong dominating set if D is a strong dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality of chromatic strong dominating set is called chromatic strong domination number of G and is denoted by $\gamma_s^c(G)$. In this paper we study relationship between strong domination and chromatic strong domination of a graph.

Keywords: Domination, Strong domination,Chromatic strong domination,Strong domination number, Chromatic number.

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1 Introduction

Let $G = (V, E)$ be finite simple, undirected graph. For $D \subseteq V$, the subgraph induced by D is denoted by $< D >$. The degree of a vertex v in a graph G is the number of edges of G incident with v and is denoted by $deg(v)$. The maximum and the minimum degrees of the vertices of G are respectively denoted by $\Delta(G)$ and $\delta(G)$. A vertex of degree 0 in G is called an isolated vertex, and a vertex of degree 1 is called a pendant vertex or an end vertex of G. Any vertex adjacent to a pendent vertex is called a support. A subset $D \subseteq V$ is said to be a strong dominating set if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E$ and $deg(u) \geq deg(v)$. The minimum cardinality of a strong dominating set is called the strong domination number of G and is denoted by $\gamma_s(G)$. The strong dominating set of cardinality $\gamma_s(G)$ is denoted by γ_s-set of G. A colouring of a graph G is an assignment of colours to all its vertices. A proper colouring is a colouring in which adjacent vertices are assigned different colours. The chromatic number $\chi(G)$ is the minimum number of colors necessary to give a proper colouring of G. A set $D \subseteq V$ is said to be a chromatic strong dominating set if D is a strong dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality of a chromatic strong dominating set is called the chromatic strong domination number of G and is denoted by $\gamma_s^c(G)$. The chromatic strong dominating set of cardinality $\gamma_s^c(G)$ is denoted by γ_s^c-set of G. By the definition, every chromatic strong dominating set is a strong dominating set. But the converse is not true. In some graphs, every γ_s-set is a chromatic strong dominating set. In some graphs, some γ_s-set is a chromatic strong dominating set and some γ_s-set is not a chromatic strong. dominating set while in some graphs no γ_s-set is a Chromatic Strong dominating set.

By observing these graphs with these properties, we wish to classify all graphs with these properties. By the motivation of these observative, we define that, a graph G is in $(\gamma_s \equiv \gamma_s^c)$-class if every γ_s-set is a chromatic strong dominating set. If some γ_s-set is a chromatic strong dominating set and some γ_s-set is not a chromatic strong dominating set, we say that G is in $(\gamma_s \cong \gamma_s^c)$-class. If no γ_s-set is a chromatic strong dominating set, we say that G is in $(\gamma_s \neq \gamma_s^c)$-class. For example, $K_{n,n}$ is in $(\gamma_s \equiv \gamma_s^c)$-class, $K_{m,n}$, $(m \neq n)$ is in $(\gamma_s \cong \gamma_s^c)$-class and subdivision graph of $K_{1,n}$ is in $(\gamma_s \neq \gamma_s^c)$-class.

2 Equality of sets

Definition 2.1. *A graph G is said to be in $(\gamma_s \equiv \gamma_s^c)$-class if every γ_s-set is a chromatic strong dominating set.*

Definition 2.2. *A graph G is said to be in $(\gamma_s \cong \gamma_s^c)$-class if some γ_s-sets are chromatic strong dominating set and some γ_s-sets are not chromatic strong dominating set.*

Definition 2.3. *A graph G is said to be in $(\gamma_s \neq \gamma_s^c)$-class if no γ_s-set is chromatic strong dominating set.*

Remark 2.4. *Let G be a χ-critical graph with $|V(G)| > 1$. Then G is in $(\gamma_s \neq \gamma_s^c)$-class. (Since for such graphs $\gamma_s(G) < n$ and $\gamma_s^c(G) = n$)*

Theorem 2.5. *Let P_n be a path on n vertices. Then*

1. *The path P_n belongs to $(\gamma_s \neq \gamma_s^c)$-class if and only if $n \equiv 0$ or $2\ (mod\ 3)$.*

2. *The path P_n belongs to $(\gamma_s \cong \gamma_s^c)$-class if and only if $n \equiv\ 1\ (mod\ 3)$*

Proof. Let $P_n : u_1, u_2, ..., u_n$ be a path on n vertices.
Case(i) Let $n = 3k$.

Then $\gamma_s(P_{3k}) = k$ and it has a unique γ_s-set $D = \{u_2, u_5, u_8, ..., u_{3k-1}\}$. Since D is independent, $\chi(< D >) = 1$. But $\chi(G) = 2$. Therefore D is not a chromatic strong dominating set. Hence P_{3k} belongs to $(\gamma_s \neq \gamma_s^c)$-class.
Case(ii) Let $n \equiv 1\ (mod\ 3)$. Let $n = 3k + 1$. Then $\gamma_s(P_{3k+1}) = k + 1$. Let $D_1 = \{u_2, u_5, u_8, ..., u_{3k-1}, u_{3k+1}\}$ and $D_2 = \{u_2, u_5, u_8, ..., u_{3k-1}, u_{3k}\}$. Then D_1 and D_2 are γ_s- sets. But D_1 is independent and D_2 is not independent. Therefore D_1 is not a chromatic strong dominating set and D_2 is a γ_s^c- set. Thus, P_{3k+1} is in $(\gamma_s \cong \gamma_s^c)$-class.
Case(iii) Let $n \equiv 2\ (mod\ 3)$. Let $n = 3k + 2$. Then $\gamma_s(P_{3k+2}) = k + 1$. Let D be a γ_s-set. Suppose D is not independent . Let $v_i, v_{i+1} \in D$.
Subcase (i) Let $i = 3l$. Consider the path $P_1 : u_1, u_2, ..., u_{3l-2}$ and the path $P_2 : u_{3l+3}, ..., u_{3k+2}$. Then $\gamma_s(P_1) = \lceil \frac{3l-2}{3} \rceil = l$ and $\gamma_s(P_2) = \left\lceil \frac{3(k-l)}{3} \right\rceil = k - l$. Therefore $\gamma_s(P_n) = \gamma_s(P_1) + 2 + \gamma_s(P_2) = l + 2 + k - l = k + 2$, which is a contradiction.
Subcase (ii) Let $i = 3l + 1$. Consider the path $P_1 : u_1, u_2, ..., u_{3l-1}$ and the path $P_2 : u_{3l+4}, ..., u_{3k+2}$. Then $\gamma_s(P_1) = \lceil \frac{3l-1}{3} \rceil = l$ and $\gamma_s(P_2) = \left\lceil \frac{3(k-l)-1}{3} \right\rceil = k - l$. Therefore $\gamma_s(P_n) = \gamma_s(P_1) + 2 + \gamma_s(P_2) = l + 2 + k - l = k + 2$, which is a contradiction.
Subcase (iii) Let $i = 3l + 2$. Consider the path $P_1 : u_1, u_2, ..., u_{3l}$ and the path $P_2 : u_{3l+5}, ..., u_{3k+2}$. Then $\gamma_s(P_1) = \lceil \frac{3l}{3} \rceil = l$ and $\gamma_s(P_2) = \left\lceil \frac{3(k-l)-2}{3} \right\rceil = k - l$. Therefore $\gamma_s(P_n) = \gamma_s(P_1) + 2 + \gamma_s(P_2) = l + 2 + k - l = k + 2$, which is a contradiction. Therefore D is independent. Hence, D is not chromatic strong dominating set. Therefore P_{3k+2} belongs to $(\gamma_s \neq \gamma_s^c)$- class. Hence, the theorem. ∎

Theorem 2.6. *The cycle C_n of $(\gamma_s \neq \gamma_s^c)$-class if and only if n is odd or $n \equiv 0$ or $2\ (mod\ 3)$, when n is even.*

Proof. Let $C_n : u_1, u_2, u_3, ..., u_n$ be a cycle on n- vertices.

Case(i) Let n be odd. $\gamma_s(C_n) = \lceil \frac{n}{3} \rceil < n = \gamma_s^c(C_n)$ when n is odd. Thus, no γ_s-set is a chromatic strong dominating set. Hence C_n of $(\gamma_s \neq \gamma_s^c)$-class.

Cass(ii): Let n be even and $n \equiv 0\ (mod\ 3)$. Let $n = 3k$, let D be a γ_s-set. Then $D_1 = \{u_2, u_5, u_8, ..., u_{3k-1}\}$, $D_2 = \{u_3, u_6, u_9, ..., u_{3k}\}$ and $D_3 = \{u_1, u_4, u_7, ..., u_{3k-2}\}$ are the only γ_s-sets. But all D_i' are independent. Therefore $\chi(< D_i >) = 1 \neq 2 = \chi(C_n)$ for all i. Thus, C_n is of $(\gamma_s \neq \gamma_s^c)$-class.

Case(iii) Let n be even and $n \equiv 2\ (mod\ 3)$. Let $n = 3k + 2$. Then $\gamma_s(C_n) = \lceil \frac{n}{3} \rceil = \lceil \frac{3k+2}{3} \rceil = k + 1$. Let D be a γ_s-set. Suppose D is not independent. Let $v_i, v_{i+1} \in D$. Consider the path $P_1 : v_{i+3}, v_{i+4}, ..., v_n, v_1, v_2, v_3, ..., v_{i-2}$ of length $n-4$. Therefore $\gamma_s(P_1) = \lceil \frac{n-4}{3} \rceil = \lceil \frac{3k+2-4}{3} \rceil = \lceil \frac{3k-2}{3} \rceil = k$.

Therefore $\gamma_s(C_n) = \gamma_s(P_1) + 2 = k + 2$, which is a contradiction. Therefore D is independent. Therefore $\chi(< D >) = 1 \neq \chi(C_n) = 2$. Therefore every γ_s-set is not a chromatic strong dominating set. Therefore C_n is in $(\gamma_s \neq \gamma_s^c)$-class.

Case(iv) Let $n \equiv 1\ (mod\ 3)$. Let $n = 3k + 1$. Then $\gamma_s(C_{3k+1}) = k + 1$. Let $D_1 = \{u_2, u_5, u_8, ..., u_{3k-1}, u_{3k+1}\}$ and D_1 is independent.

Thus, $\chi(< D_1 >) = 1$. But $\chi(G) = 2$. Therefore D_1 is not chromatic strong dominating set. Let $D_2 = \{u_2, u_5, ..., u_{3k-1}, u_{3k}\}$ and D_2 is not independent. Thus, $\chi(< D_2 >) = 2$ and $\chi(G) = 2$. Therefore D_2 is a chromatic strong dominating set. Thus C_n is in $(\gamma_s \cong \gamma_s^c)$-class. ∎

Corollary 2.7. *C_n is of $(\gamma_s \cong \gamma_s^c)$-class if and only if n is even and $n \equiv 1\ (mod\ 3)$.*

Proposition 2.8. *Let $G = K_{m,n}$, $m \neq n$. Then G is of $(\gamma_s \neq \gamma_s^c)$-class.*

Proof. Let $G = K_{m,n}$ and $m < n$. Let (V_1, V_2) be the bipartition of $V(G)$ and $|V_1| < |V_2|$. Then V_1 is the unique γ_s-set in G and V_1 is a independent. Therefore $\chi(< V_1 >) = 1 \neq \chi(G)$, which means V_1 is not a γ_s^c-set. Thus, G is in $(\gamma_s \neq \gamma_s^c)$-class. ∎

Proposition 2.9. *Let $G = K_{n,n}$ where $m = n \geq 3$. Then G is of $(\gamma_s \equiv \gamma_s^c)$- class.*

Proof. Let $G = K_{n,n}$ where $n \geq 3$. Let (V_1, V_2) be the bipartition of $V(G)$ where $u \in V_1$ and $v \in V_2$. Then every γ_s-set of $K_{n,n}$, $n \geq 3$ is $\{u, v\}$, where $uv \in E(K_{n,n})$. Therefore $\gamma_s = 2$ and $\chi(< \{u, v\} >) = 2 = \chi(G)$. That is $\{u, v\}$ is a

γ_s-set. Thus every γ_s-set is γ_s^c-set. Therefore G is in $(\gamma_s \equiv \gamma_s^c)$-class. ∎

Observation 2.10. *Let $G = K_{1,n-1}$, $n \geq 3$ then G is in $(\gamma_s \neq \gamma_s^c)$-class.*

Proof. Let $G = K_{1,n-1}$. Then $\gamma_s(K1, n-1) = 1$. Let u be a full degree vertex of G. Then $D = \{u\}$ is a unique γ_s-set. But $\chi(< D >) = 1 \neq 2 = \chi(G)$. Therefore D is not a chromatic strong dominating set. Therefore G is in $(\gamma_s \neq \gamma_s^c)$-class. ∎

Proposition 2.11. *Let $G = K_n$, $n \geq 2$. Then G is in $(\gamma_s \neq \gamma_s^c)$-class.*

Proof. Let $G = K_n$, $n \geq 2$. Then $\gamma_s(K_n) = 1$. Therefore γ_s-sets are singleton sets. Therefore $\chi(< D >) = 1 \neq n = \chi(G)$ for every γ_s-set. Thus, D is not a chromatic strong dominating set. Hence G is in $(\gamma_s \neq \gamma_s^c)$-class. ∎

Theorem 2.12. *Let T be a tree. If there exist support vertices u, v such that $d(u_i) = 2$ for all u_i on the path joining u and v and $d(u, v) \equiv 0\ (mod\ 3)$. Then u, v belong to every γ_s-set of T.*

Proof. Let u, v be a pair of vertices in T such that $d(u, v) \equiv 0\ (mod\ 3)$ and $d(u_i) = 2$ for all u_i on the path joining u and v. Suppose there is a γ_s-set D such that $u \notin D$ or $v \notin D$. Let $u \notin D$. Then $N(u)$ has exactly one pendent vertex, say w. Then $w \in D$. Let $u = u_0, u_1, u_2, u_3, \cdots, u_r = v$. Then $r \equiv 0\ (mod\ 3)$. Since D is a γ_s-set and $u_0 = u \notin D$, assume that $u_2, u_5, \cdots, u_{r-1} \in D$. Let $r = 3k$. Then $|\{u_2, u_5, \cdots, u_{r-1}\}| = k$. If $v \in D$, then $D_1 = (D - \{w, u_2, u_5, \cdots, u_{r-1}\}) \bigcup \{u, u_3, u_6, \cdots, u_{r-3}\}$ is a strong dominating set and $|D_1| = |D| - 1$, which is a contradiction. If $v \notin D$, then $N(v)$ has exactly one pendent vertex x, and $x \in D$. $D_2 = (D - \{x, w, u_2, u_5, \cdots, u_{r-1}\}) \bigcup \{u, v, u_3, u_6, \cdots, u_{r-3}\}$ is a strong dominating set and $|D_2| = |D| - 1$, which is a contradiction. Hence, $u \in D$ and $v \in D$. ∎

Proposition 2.13. *Let P be the set of all pendent vertices of a tree T. If T has two vertices u, v such that $|N(u) \bigcap P| \geq 2$, $|N(v) \bigcap P| \geq 2$, and $uv \in E(T)$, Then T belongs to $(\gamma_s \equiv \gamma_s^c)$-class.*

Proof. Let $|N(u) \bigcap P| \geq 2$ and $|N(v) \bigcap P| \geq 2$. Then u, v belong to every γ_s-set D of T. Since $uv \in E(T)$, $\chi(< D >) = 2$, D is a γ_s^c-set.

Hence T is in $(\gamma_s \equiv \gamma_s^c)$-class. ■

Theorem 2.14. *Let T be a tree in which $d(u,v) \equiv 0\ (mod\ 3)$ for some support vertices and $d(u_i) = 2$ for all u_i on the path joining u and v. If there exists a support vertex w such that $vw \in E(T)$, and either $|N(w) \bigcap P| \geq 2$ or for some support vertex x, $d(w,x) \equiv 0\ (mod\ 3)$ and $d(x_i) = 2$ for all vertices x_i on the path joining w and x, then T belongs to $(\gamma_s \equiv \gamma_s^c)$-class.*

Proof. Let T be a tree in which for some support vertices u, v and w, $d(u,v) \equiv 0\ (mod\ 3)$, $d(u_i) = 2$ for all vertices u_i on the path joining u, v and $vw \in E(T)$. Suppose that either $|N(w) \bigcap P| \geq 2$ or there exists a support vertex x such that $d(w,x) \equiv 0\ (mod\ 3)$ and $d(x_i) = 2$ for all vertices x_i on the path joining w and x. Let D be a γ_s-set. By Proposition 2.13, $u, v \in D$. If $|N(w) \bigcap P| \geq 2$, then $w \in D$. If there exists a support vertex x such that $d(w,x) \equiv 0\ (mod\ 3)$ and $d(x_i) = 2$ for all vertices x_i on the path joining w and x, then by Proposition 2.13, $w, x \in D$. Thus, in both cases, $w \in D$. Since $vw \in E(T)$, $\chi(< D >) = 2$. Thus, D is a γ_s^c-set. Hence T is in $(\gamma_s \equiv \gamma_s^c)$-class. ■

Theorem 2.15. *Let D be a caterpillar. T is in $(\gamma_s \equiv \gamma_s^c)$-class if and only if T has two support vertices u, v satisfying one of the following properties:*
(i). $|N(u) \bigcap P| \geq 2$, $|N(v) \bigcap P| \geq 2$ and $uv \in E(T)$
(ii). If $d(u,v) \equiv 0\ (mad\ 3)$ and $d(u_i) = 2$ for all u_i on the path joining u and v, there exists a support vertex w such that $vw \in E(T)$ and $|N(w) \bigcap P| \geq 2$ or for some support vertex x, $d(w,x) \equiv 0\ (mad\ 3)$ and $d(x_i) = 2$ for all vertices x_i on the path joining w and x.

Proof. Let T be a caterpillar. Assume that T is in $(\gamma_s \equiv \gamma_s^c)$-class. Suppose that for every pair u, v of support vertices, the given two conditions fail. Let D be a γ_s-set with minimum number of edges in $< D >$. Since D is in $(\gamma_s \equiv \gamma_s^c)$-class, $\chi(< D >) = 2$. Therefore there exist $u, v \in D$ such that $uv \in E(T)$. Then $u, v \in V(T) - P$. Since $uv \in E(T)$ and since (i) is not true, $|N(u) \bigcap P| \leq 1$ or $|N(v) \bigcap P| \leq 1$.
Case (i) Let $N(u) \bigcap P = \phi$. Since D is a γ_s-set and $uv \in E(T)$, $u \notin P$. Since T is a caterpillar and $N(u) \bigcap P = \phi$, $d(u) = 2$. Let $x \in N(u) - \{v\}$. Since D is a γ_s-set and $u, v \in D$, $(N(x) - \{u\}) \bigcap D = \phi$. Then $D_1 = (D - \{u\}) \bigcup \{x\}$ is a

γ_s-set and $|E(< D_1 >)| = |E(< D >)| - 1$ which is a contradiction to the fact that D is a γ_s-set with minimum number of edges in $< D >$.

Case (ii) Let $N(u) \bigcap P = \{x\}$ and $N(v) \bigcap P = \{y\}$. If $(N(u) - \{v\}) \bigcap D \neq \phi$, then $D_2 = (D - \{u\}) \bigcup \{x\}$ is a γ_s-set with $|E(< D_2 >)| = |E(< D >)| - 1$ which is a contradiction. Let $(N(u) - \{v\}) \bigcap D = \phi$ and $(N(v) - \{u\}) \bigcap D = \phi$. Let $w_1 \neq v$ be a support vertex such that $d(u, w_1)$ is minimum and $d(u_i) = 2$ for all vertices u_i on the path joining u and w. Let $w_2 \neq u$ be a support vertex such that $d(v, w_2)$ is minimum and $d(v_i) = 2$ for all vertices v_i on the path joining v and w. Let $d(u, w_1) \equiv 0 \ (mod\ 3)$. Since $uv \in E(T)$, by our assumption, $d(v, w_2) \equiv 1 \ (mod\ 3)$. Let $v = v_0, v_1, v_2, \cdots, v_r = w_2$ and $d(v_i) = 2$ for all $i = 1, 2, \cdots, r-1$. Then $r = 3k+1$ or $3k+2$. Since D is a γ_s-set and $v \in D$ we can assume that $v_3, v_6, v_9, \cdots, v_{r-1} \in D$ if $r = 3k+1$ or $v_3, v_6, \cdots, v_{r-2} \in D$ if $r = 3k+2$. Let $w_2 \in D$. If $r = 3k+1$, then $D_3 = (D - \{v, v_3, v_6, \cdots, v_{r-1}\}) \bigcup \{v_2, v_5, \cdots, v_{r-2}\}$ is a γ_s-set. But $|E(< D_3 >)| = |E(< D >)| - 2$, which is a contradiction. If $r = 3k+2$, then $D_4 = (D - \{v, v_3, v_6, \cdots, v_{r-2}\}) \bigcup \{y, v_2, v_5, \cdots, v_{r-3}\}$ is a γ_s-set. But $|E(< D_4 >)| = |E(< D >)| - 1$, which is a contradiction. If $w_2 \notin (D)$, then $N(w_2) \bigcap P = \{z\}$ and $z \in D$. If $r = 3k+1$, then $D_5 = (D - \{v, z, v_3, v_6, \cdots, v_{r-1}\}) \bigcup \{y, w_2, v_2, v_5, \cdots, v_{r-2}\}$ is a γ_s-set and $|E(< D_5 >)| = |E(< D >)| - 1$, which is a contradiction. If $r = 3k+2$, then $D_6 = (D - \{v, z, v_3, v_6, \cdots, v_{r-2}\}) \bigcup \{y, w_2, v_2, v_5, \cdots, v_{r-3}\}$ is a γ_s-set and $|E(< D_6 >)| = |E(< D >)| - 1$, which is a contradiction. Thus in all cases, we get contradictions. Hence one of the two conditions (i), (ii) holds.

Conversely, assume that T has two support vertices u, v satisfying one of the given two conditions. By proposition 2.13, and by theorem 2.14, T is in $(\gamma_s \equiv \gamma_s^c)$-class. ∎

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