International Journal of Mathematics and Soft Computing
Vol.1, No.1 (2011), 55 - 67
ISSN 2249 - 3328

Point set domination with reference to degree

R.Poovazhaki
Department of Mathematics, EMG Yadava Women's College,
Madurai 625 014, India.
E-mail: rpoovazhaki@yahoo.co.in

V. Swaminathan
Ramanujan Research Center, Department of Mathematics,
Saraswathi Narayanan College, Madurai - 625 022, India.
E-mail: sulanesri@yahoo.com

Abstract

E.Sampathkumar et al introduced [7] the concept of point set domination number of a graph. A set $D \subseteq V(G)$ is said to be a point set dominating set (psd set), if for every $S \subseteq V - D$ there exists a vertex $u \in D$ such that the subgraph $\langle S \cup \{u\} \rangle$ induced by $S \cup \{u\}$ is connected. The minimum cardinality of a psd set is called the point set domination number of G and is denoted by $\gamma_p(G)$.In this paper psd sets are analysed with respect to the strong [9] domination parameter for separable graphs. The characterization of separale graphs with equal psd number and spsd number is derived.

Key words: separable graph, point set domination, strong point set domination

AMS Subject Classification(2010): 05C69

1 Introduction

A set $D \subseteq V(G)$ is said to be a strong point set dominating set (spsd set), if for every $S \subseteq V - D$ there exists a vertex $u \in D$ such that the subgraph $\langle S \cup \{u\} \rangle$ induced by $S \cup \{u\}$ is connected and $d(u) \geq d(s)$ for all $s \in S$ where $d(u)$ denote the degree of the vertex u. The minimum cardinality of an spsd set is called the strong point set domination number of G and is denoted by $\gamma_{sp}(G)$. A connected graph with atleast one cut vertex is called a separable graph. If B is a block of a separable graph G with psd set B', then $(V - B) \cup B'$ is a psd set of G but need not be an spsd set of G as seen in the following discussion. Hence the spsd sets of G are characterized first and then analysed with reference to the spsd sets of the blocks of G. The characterization of separale graphs with equal psd number and spsd number is derived. In the following discussion, a graph G always means a connected graph.

2 Main Results

Definition 2.1. *A set $D \subset V(G)$ is said to be a strong point set dominating set (spsd set) of G if for every $S \subseteq V - D$ there exists a vertex $u \in D$ such that the subgraph $\langle S \cup \{u\} \rangle$ induced by $S \cup \{u\}$ is connected and $d(u) \geq d(s)$ for all $s \in S$ where $d(u)$ denote the degree of the vertex u.*
The minimum cardinality of an spsd set is called the strong point set domination number of G and is denoted by $\gamma_{sp}(G)$.

Proposition 2.2. *A subset D of V is an spsd set if and only if for every independent set $S \subseteq V - D$ there exists $u \in D$ such that $S \subseteq N(u)$ and $d(u) \geq d(s)$ for all $s \in S$*

Proof. If D is an spsd set of G, then the condition follows from the definition of D. Conversely, suppose the given condition is satisfied. Let $S \subseteq V - D$ be any set. If S is independent, then by the given condition there exists $u \in D$ such that $\langle S \cup \{u\} \rangle$ is connected and $d(u) \geq d(s)$ for all $s \in S$. If S is not independent, then let $S = S_1 \cup S_2$ where S_1 is a maximal independent subset of S. Let $s' \in S$ be such that $d(s') = Max_{s \in S} \{d(s)\}$.
Case (i) $s' \in S_1$.
$S_1$is a maximal independent subset of S implies there exists $u \in D$ such that $S_1 \subseteq N(u)$ and $d(u) \geq d(s)$ for all $s \in S_1$. Therefore, $d(u) \geq d(s')$. S_1 is maximal independent subset of S implies every vertex of S_2 is adjacent to at least one vertex in S_1. Hence $\langle S_1 \cup S_2 \cup \{u\} \rangle$ is connected. Also $d(u) \geq d(s')$ implies $d(u) \geq d(s') \geq d(s)$ for all $s \in S$. Hence $\langle S \cup \{u\} \rangle$ is connected and $d(u) \geq d(s)$ for all $s \in S$.
Case (ii) $s' \in S_2$.
$s' \in S_2$ implies that s' is adjacent to at least one vertex in S_1.
(ii) - (a): s' is adjacent to all vertices in S_1.
Every vertex of S_2 is adjacent to at least one vertex of S_1. Therefore, $\langle S_1 \cup S_2 \rangle$ is connected. Also $s' \in V - D$ implies there exists $u \in D$ such that $us' \in E(G)$ and $d(u) \geq d(s')$. Therefore, $\langle S \cup \{u\} \rangle$ is connected and $d(u) \geq d(s') \geq d(s)$ for all $s \in S$.
(ii) - (b): There are vertices in S_1 which are not adjacent to s'.
Let $A = \{s \in S_1 / s \notin N(s')\} \cup \{s'\}$

Then A is independent and therefore there exists $u \in D$ such that $\langle A \cup \{u\} \rangle$ is connected and $d(u) \geq d(a)$ for all $a \in A$. By the definition of A, $s' \in A$. Therefore, $\langle S \cup \{u\} \rangle$ is connected and $d(u) \geq d(s') \geq d(s)$ for all $s \in S$. Hence D is an spsd set of G. ∎

Remark 2.3. *In the remaining discussion of this paper, a graph G always means a separable graph.*

Observation 2.4. *If B is a block with spsd set B', then $(V - B) \cup B'$ need not be an spsd set of G.*

Proof. Consider the following figure:

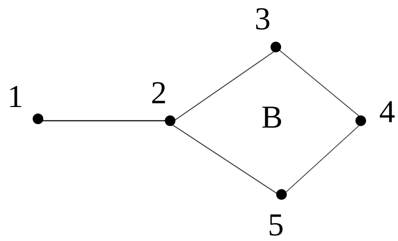

Figure 1

$B' = \{3, 5\}$ is an spsd set of B. Then $(V - B) \cup B' = \{1, 3, 5\}$ is a psd set of G but not an spsd set of G since $d(2) > d(1), d(3), d(5)$. ∎

Remark 2.5. *If a block B has an spsd set B' containing all cut vertices belonging to B, then $(V - B) \cup B'$ is an spsd set of G.*

Proof. Let $S \subseteq V - [(V - B) \cup B']$ be independent. Then $S \subseteq B - B'$. B' is an spsd set of B implies there exists $u \in B'$ such that $S \subseteq N_B(u)$ and $d_B(u) \geq d_B(s)$ for all $s \in S$. Since B' contain all cut vertices belonging to B, $d_B(s) = d_G(s)$ for all $s \in S$. Hence $d_G(u) \geq d_B(u) \geq d_B(s) = d_(s)$ for all $s \in S$. That is, $S \subseteq N(u)$ and $d_G(u) \geq d_G(s)$ for all $s \in S$. Therefore, $(V - B) \cup B'$ is an spsd set of G. ∎

Therefore, separable graphs in which every block has a γ_{sp} set containing all cut vertices belonging to B are considered in the following discussion.

Definition 2.6. $k_{sp} = Max_{B \in B_G} \{|B| - \gamma_{sp}(B)\}$ *where B_G denote the set of all blocks of G.*

Remark 2.7.

(i) $\gamma_{sp}(G) \leq n - k_{sp}$.

(ii) $\gamma_{sp}(G) \leq n - \Delta$.

Proof.
(i) : $(V - B) \cup B'$ is an spsd set of G implies $\gamma_{sp}(G) \leq n - (|B - B'|)$. Choose a block B for which $|B - B'| = k_{sp}$. Hence $\gamma_{sp}(G) \leq n - k_{sp}$.
(ii) : $D = V(G) - N(u)$ where $d(u) = \Delta$ is an spsd set of G and hence $\gamma_{sp}(G) \leq n - \Delta$. ∎

Remark 2.8. *If D is an γ_{sp} set of a separable graph G, then there are three cases:*

(i) $V - D$ *contain vertices of different blocks.*

(ii) $V - D \subset B$.

(iii) $V - D = B$ *for some block* B.

Definition 2.9. *When $V - D \subset B$, define*
$P(B, D) = \{u \in V - D/N(u) \cap (B \cap D) = \phi\}$.

Remark 2.10. *If $P(B, D) \neq \phi$, then $\gamma_{sp}(G) = n - \Delta$.*

Remark 2.11. *$B \cap D$ is an spsd set of $B - P(B, D)$.*

Proof. Let $S \subseteq B - P(B, D) - B \cap D$ be an independent subset. Then $S \subseteq V - D$. Therefore, there exists $u \in D$ such that $T \subseteq N(u)$ and $d(u) \geq d(s)$ for all $s \in S$. S is an independent subset implies u is adjacent to more than one vertex in B and hence $u \in B \cap D$.
Case (i) u is not a cut vertex.
Then $d_G(u) = d_B(u)$. Hence $d_B(u) = d_G(u) \geq d_G(s) \geq d_B(s)$ for all $s \in S$. That is, there exists $u \in B \cap D$ such that $S \subseteq N(u)$ and $d_B(u) \geq d_B(s)$ for all $s \in S$. That is, $B \cap D$ is an spsd set of $B - P(B, D)$.
Case(ii) u is a cut vertex.
Then every path connecting a point of $V - D$ to a point of $D - B \cap D$ must contain u. Hence $N(s) \cap (D - B \cap D) = \phi$ for all $s \in S$. Therefore, $d_G(s) = d_B(s)$ for all $s \in S$. If there exists no $x \in B \cap D$ such that $S \subseteq N(x)$ with $d_B(x) \geq d_B(s)$ for all

$s \in S$, then as $N(s) \cap (D - B \cap D) = \phi$ there exists no $x \in D$ such that $S \subseteq N(x)$ and $d_G(x) \geq d_B(x) \geq d_B(s) = d_G(s)$ for all $s \in S$ which is a contradiction to the fact that D is an spsd set of G.
Hence there exists $x \in B \cap D$ such that $S \subseteq N(x)$ and $d_B(x) \geq d_B(s)$ for all $s \in S$. That is, $B \cap D$ is an spsd set of $B - P(B, D)$. ■

Remark 2.12. *If $P(B, D) = \phi$, then $B \cap D$ is an spsd set of B.*

Remark 2.13. *If $P(B, D) = \phi$, then $\gamma_{sp}(G) = n - k_{sp}$.*

Proof. $P(B, D) = \phi$ implies $B \cap D$ is an spsd set of B and hence $\gamma_{sp}(B) \leq |B \cap D|$. Also $\gamma_{sp}(B) \geq |B \cap D|$. For, if $\gamma_{sp}(B) > |B \cap D|$, then $(V - B) \cup B'$ is an spsd set of G where $|B'| = \gamma_{sp}(B)$. Then $|D| = |(V - B) \cup (B \cap D)| > |(V - B) \cup B'|$. That is, there exists an spsd set $(V - B) \cup B'$ of G with cardinality less than $|D|$ which is a contradiction. Hence $\gamma_{sp}(B) \geq |B \cap D|$.

Therefore, $\gamma_{sp}(B) = |B \cap D|$. Hence, $\gamma_{sp}(G) = |D| = |(V - B) \cup (B \cap D)| = |(V - B) \cup B'| \geq n - k_{sp}$. Therefore, $\gamma_{sp}(G) = n - k_{sp}$. ■

Remark 2.14. *If $V - D = B$ for some block B, then $\gamma_{sp}(G) = n - \Delta$.*

Proof. $V - D = B$ implies $\langle V - D \rangle$ is complete. Therefore, $d(u) \geq |V - D|$ and hence $|D| \geq n - d(u) \geq n - \Delta$ for any vertex $u \in V - D$. Therefore, $\gamma_{sp}(G) = n - \Delta$. ■

Theorem 2.15. *If G is a connected graph with cut vertices, then*
$\gamma_{sp}(G) = Min\{n - \Delta, n - k_{sp}\}$

Proof. Let D be a minimum spsd set of G. Then $|D| = \gamma_{sp}(G)$.
Case (i) $V - D$ contain vertices of different blocks.
Then $V - D \subseteq N(w)$. $d(w) \geq |V - D|$ implies $|D| \geq n - d(w) \geq n - \Delta$. Hence $|D| \geq n - \Delta$. Therefore, $|D| = n - \Delta$. That is, $\gamma_{sp}(G) = n - \Delta$. Hence $n - \Delta = \gamma_{sp}(G) \leq n - k_{sp}$. That is, $\gamma_{sp}(G) = Min\{n - \Delta, n - k_{sp}\}$.
Case (ii) $V - D \subset B$ for some block B.
Then if $P(B, D) \neq \phi$, then $\gamma_{sp}(G) = n - \Delta$. Hence $n - \Delta = \gamma_{sp}(G) \leq n - k_{sp}$. That is, $\gamma_{sp}(G) = Min\{n - \Delta, n - k_{sp}\}$.

If $P(B, D) = \phi$, then $\gamma_{sp}(G) = n - k_{sp}$.
Hence $n - k_{sp} = \gamma_{sp}(G) \leq n - \Delta$. That is, $\gamma_{sp}(G) = Min\,\{n - \Delta, n - k_{sp}\}$.
Case (iii) $V - D = B$ for some block B.
Then $\gamma_{sp}(G) = n - \Delta$. Hence $n - \Delta = \gamma_{sp}(G) \leq n - k_{sp}$. That is, $\gamma_{sp}(G) = Min\,\{n - \Delta, n - k_{sp}\}$. Hence in all cases
$\gamma_{sp}(G) = Min\,\{n - \Delta, n - k_{sp}\}$. ∎

Theorem 2.16. *$k = k_{sp}$ if and only if there exists a block B such that $k = |B| - \gamma_p(B)$ and $\gamma_p(B) = \gamma_{sp}(B)$.*

Proof. Let $k = k_{sp}$.
If B is a block with $k_{sp} = |B| - \gamma_{sp}(B)$, then $\gamma_p(B) = \gamma_{sp}(B)$. For, if $\gamma_p(B) \neq \gamma_{sp}(B)$, then $\gamma_p(B) < \gamma_{sp}(B)$. Therefore, $k_{sp} = |B| - \gamma_{sp}(B) < |B| - \gamma_p(B) \leq k$. This implies $k_{sp} < k$ which is a contradicts $k = k_{sp}$). Hence for any block B for which $k_{sp} = |B| - \gamma_{sp}(B)$, $\gamma_p(B) = \gamma_{sp}(B)$. If $k = k_{sp}$, then $k = |B| - \gamma_{sp}(B) = |B| - \gamma_p(B)$. Hence there exists a block for which $k = |B| - \gamma_p(B)$ and $\gamma_p(B) = \gamma_{sp}(B)$.
Conversely, let there exists a block B such that $k = |B| - \gamma_p(B)$ and $\gamma_p(B) = \gamma_{sp}(B)$, $k = |B| - \gamma_p(B) = |B| - \gamma_{sp}(B) \leq k_{sp}$. Hence $k \leq k_{sp}$,. . . ,(1). For any block B, $\gamma_p(B) \leq \gamma_{sp}(B)$.
This implies $k = (|B| - \gamma_p(B)) \geq |B| - \gamma_{sp}(B)$. Choose a block B for which $k_{sp} = |B| - \gamma_{sp}(B)$. Then $k = |B| - \gamma_p(B) = k_{sp}$,. . . ,(2). (1) and (2) together give $k = k_{sp}$. ∎

Notation 2.17.

(i) $D_{sp}(G)$ *denotes the set of all spsd sets of* G.

(ii) $D_{sp}(G; X_1)$ *denotes the set of all spsd sets D of G with $V - D \subset B$ and $P(B, D) = \phi$ for some $B \in B_G$.*

(iii) $D_{sp}(G; X_1)$ *denotes the set of all spsd sets D of G with $V - D \subset B$ and $P(B, D) \neq \phi$ for some $B \in B_G$.*

(iv) $D_{sp}(G; X_1)$ *denotes the set of all spsd sets D of G with $V - D = B$.*

Theorem 2.18. *For any separable graph $D_{sp}(G; X_1) \neq \phi$.*

Proof. For every block B there exists a γ_{sp} set B' containing all cut vertices belonging to B. Let $D = (V - B) \cup B'$. Then $V - D = B - B' \subset B$ and $B \cap D = B'$. $D \in \boldsymbol{D}_{sp}(G)$ and $B - B'$ has no cut vertices. Therefore, $N(u) \cap (D - (B \cap D)) = N(u) \cap (V - B) = \phi$ for all $u \in B - B'$. Hence $N(u) \cap (D - B') = \phi$ and $N(u) \cap (B \cap D) = N(u) \cap B' \neq \phi$ for all $u \in V - D$. Therefore, $P(B, D) = \phi$. Hence $V - D \subset B$ with $P(B, D) = \phi$. That is, $D \in \boldsymbol{D}_{sp}(G; X_1)$. Therefore, $\boldsymbol{D}_{sp}(G; X_1) \neq \phi$. ∎

Theorem 2.19. *$\boldsymbol{D}_{sp}(G; X_2) \neq \phi$ if and only if there exists $B \in B_G$ such that B can be partitioned into three non empty sets V_1, V_2 and V_3 satisfying the following conditions:*

(a) $\langle V_1 \rangle$ is complete, $N(x) \cap V_2 = V_2$, $N(x) \cap V_3 = \phi$ and there exists $u \in N(x) \cap (V - B)$ with $d_G(u) \geq d_G(x)$, for each $x \in V_1$.

(b) $V_1 \cup V_2 \cup V_3 = B$.

(c) $d_B(v) = d_G(v)$ for all $v \in V_2$.

(d) $V_3 \in \boldsymbol{D}_{sp}(V_2 \cup V_3)$.

Proof. (a): Let $D \in \boldsymbol{D}_{sp}(G; X_2)$.
Then there exists $B \in B_G$ such that $V - D \subset B$ and $P(B, D) \neq \phi$. Therefore, $(V - D) - P(B, D) \neq \phi$ and $B \cap D \neq \phi$. Now, let $V_1 = P(B, D)$. Then $\langle V_1 \rangle$ is complete. Let $V_2 = (V - D) - P(B, D)$ and $V_3 = B \cap D$. Then for each $x \in V_1$, $N(x) \cap V_2 = V_2$, $N(x) \cap V_3 = \phi$ and there exists $u \in D - (B \cap D)(= V - B)$ such that $ux \in E(G)$ and $d_G(u) \geq d_G(x)$.
(b): $V_1 \cup V_2 \cup V_3 = P(B, D) \cup [(V - D) - P(B, D)] \cup (B \cap D) = B$.
(c): $P(B, D) \neq \phi$ implies there exists $u \in P(B, D)$. Then $N(u) \cap (B \cap D) = \phi$ and $N(u) \cap (D - (B \cap D)) \neq \phi$. That is, u is a cut vertex.
Hence every path connecting a point of B and a point of $D - (B \cap D)$ must contain u. Therefore, for every $v \in (V - D) - P(B, D) = V_2$, $N(v) \cap (D - B \cap D) = \phi$. That is, v is not a cut vertex. Hence $d_B(v) = d_G(v)$ for all $v \in V_2$.
(d): Let $S \subset (V_2 \cup V_3) - V_3$ be independent. Then $S \subset (V_2 \cup V_3) - V_3 = V_2 = V - P(B, D) - D = (B - B \cap D) - P(B, D) \subset B - P(B, D)$. $B \cap D \in \boldsymbol{D}_{sp}(B - P(B, D))$ implies there exists $u \in B \cap D$ such that $\langle S \cup \{u\} \rangle$ is connected and $d_B(u) \geq d_B(s)$ for all $s \in S$. $s \in S \subset V_2$ implies $d_B(s) = d_G(s)$ for all $s \in S$

(by (c)). Hence $\langle S \cup \{u\} \rangle$ is connected and $d_G(u) \geq d_B(u) \geq d_B(s) = d_G(s)$ for all $s \in S$.

Conversely, suppose there exists $B \in \mathbf{B}_G$ satisfying (a), (b), (c) and (d). Then, let $D = V - V_1 \cup V_2 = (V - B) \cup V_3$.
$V - D = B \cap (V - V_3) = V_1 \cup V_2 \subset B$.
$P(B, D) = \{u \in V - D / N(u) \cap (B \cap D) = \phi\}$.
By condition (a) $P(B, D) \neq \phi$ and $V_1 \subseteq P(B, D)$.
By condition (b) $P(B, D) \cap V_2 = \phi$. $P(B, D) \subseteq V - D = V_1 \cup V_2$,
$P(B, D) = V_1$.
Claim: $D \in \mathbf{D}_{sp}(G)$.
Let $W \subset V - D$ be independent. If $W \cap P(B, D) \neq \phi$ and $W = \{w\}$, then $w \in P(B, D)$. By condition (a) there exists $u \in N(w) \cap (V - B)$ with $d_G(u) \geq d_G(w)$. That is, there exists $u \in D$ such that $w \in N(u)$ and $d_G(u) \geq d_G(w)$. If $W \cap P(B, D) = \phi$, then $W \subset V_2 = (V - D) - V_1$, $V_3 \in \mathbf{D}_{sp}(V_2 \cup V_3)$. Therefore, there exists $u \in V_3$ such that $W \subseteq N(u)$ and $d_B(u) \geq d_B(v)$ for all $v \in W$. $W \subset V_2$ implies $N(v) \cap (V - B) = \phi$ for all $v \in W$. Therefore, $d_G(v) = d_B(v)$ by condition (d) and hence $d_G(u) \geq d_B(u) \geq d_B(v) = d_G(v)$. Hence $D \in \mathbf{D}_{sp}(G)$ with $V - D \subset B$ and $P(B, D) \neq \phi$. This implies $D \in D_{sp}(G; X_2)$. ∎

Observation 2.20. *The partition of B in the above theorem is unique.*

Proof. For if, there exists another partition B_1, B_2, B_3 such that

(a) $\langle B_1 \rangle$ is complete, for each $x \in B_1$, $N(x) \cap B_2 = B_2$, $N(x) \cap B_3 = \phi$ and there exists $u \in V - B$ such that $ux \in E(G)$ and $d_G(u) \geq d_G(x)$.

(b) $B_1 \cup B_2 \cup B_3 = B$.

(c) $d_B(v) = d_G(v)$ for all $v \in B_2$.

(d) $B_3 \in \mathbf{D}_{sp}(B_2 \cup B_3)$. $D = V_3 \cup (V - B) = (V - B) \cup B_3$. Therefore, $V_3 = B_3$. If there exists $u \in V_1$ such that $u \notin B_1$, then there exists $d \in B_3$ such that $ud \in E(G)$. Hence $N(u) \cap B_3 \neq \phi$ which implies $N(x) \cap V_3 \neq \phi$ which is a contradiction. Hence $V_1 \subseteq B_1$. If there exists $u \in B_1$ such that $u \notin V_1$, then there exists $d \in V_3$ such that $ud \in E(G)$. Hence $N(u) \cap V_3 \neq \phi$ which implies $N(u) \cap B_3 \neq \phi$ with $u \in B_1$ which is a contradiction. Hence $B_1 = V_1$. Therefore, $B_2 = V_2$. That is, the partition is unique. ∎

Theorem 2.21. $\boldsymbol{D}_{sp}(G;Y) \neq \phi$ *if and only if there exists* $B \in \boldsymbol{B}_G$ *such that the following conditions are satisfied:*

(a) $\langle B\rangle$ *is complete.*

(b) *For each* $x \in B$, *there exists* $u \in N(x) \cap (V - B)$ *with* $d_G(u) \geq |B|$.

Proof. Let $D \in \boldsymbol{D}_{sp}(G;Y)$. Then there exists $B \in \boldsymbol{B}_G$ with $V - D = B$. That is, $B \cap D = \phi$. Hence $N(x) \cap (B \cap D) = \phi$ for every $x \in V - D$. $D \in \boldsymbol{D}_{sp}(G)$ implies for every $x \in V - D (= B)$ there exists $u \in N(x) \cap D$. Therefore, $P(B, D) = V - D = B$ and hence $\langle B\rangle = \langle P(B, D)\rangle$ is complete. Hence $d_G(x) \geq |B|$.
Conversely, suppose the above two conditions are satisfied. Let $D = V - B$. Then $V - D = B = P(B, D)$. Therefore, $D \in \boldsymbol{D}_{sp}(G;Y)$. ∎

Observation 2.22. $\boldsymbol{D}_{sp}(G;Z) \neq \phi$ *if* G *has a cut vertex* w *with* $d(w) \geq d(v)$ *for all* $v \in N(w)$.

Proof. If $D = V - N(w)$, then $V - D = N(w)$ contain vertices of different blocks, and $D \in \boldsymbol{D}_{sp}(G)$. Hence $D \in \boldsymbol{D}_{sp}(G;Z)$. ∎

Observation 2.23. *If* $\Delta > k_{sp}$, *then there exists a vertex* u *of degree* Δ *such that* $N(u)$ *is not contained in a single block.*

Proof. If for every vertex u of degree Δ there exists a block B such that $N(u) \subset B$, then $\Delta > k_{sp} > 1$ implies $N[u] \subset B$. Therefore, $u \in B$. Hence $B - N(u) \in \boldsymbol{D}_{sp}(B)$. But then $\gamma_{sp}(B) \leq |B| - |N(u)|$. That is, $|N(u)| \leq |B| - \gamma_{sp}(B) \leq k_{sp}$. That is, $\Delta \leq k_{sp}$ which is a contradiction. Hence if $\Delta > k_{sp}$ there exists a vertex u of degree Δ such that $N(u)$ is not contained in a single block. ∎

Notation 2.24.

(i) $\boldsymbol{D}^o_{sp}(G)$ *- denote the set of all minimum spsd sets of* G.

(ii) $\boldsymbol{D}^o_{sp}(G;X_1) = \boldsymbol{D}^o_{sp}(G) \cap \boldsymbol{D}_{sp}(G;X_1)$.

(iii) $\boldsymbol{D}^o_{sp}(G;X_2) = \boldsymbol{D}^o_{sp}(G) \cap \boldsymbol{D}_{sp}(G;X_2)$.

(iv) $\boldsymbol{D}^o_{sp}(G;Y) = \boldsymbol{D}^o_{sp}(G) \cap \boldsymbol{D}_{sp}(G;Y)$.

(v) $\boldsymbol{D}^o_{sp}(G;X_1) = \boldsymbol{D}^o_{sp}(G) \cap \boldsymbol{D}_{sp}(G;Z)$.

Remark 2.25. *The following theorems are the immediate consequences of the previous results.*

Theorem 2.26. $D^o_{sp}(G;Z) = \phi$ *if and only if one of the following two conditions is satisfied.*

(i) $\Delta < k_{sp}$.

(ii) $\Delta = k_{sp}$ *and for every vertex* u *of degree* Δ*,* $N(u) \subset B$ *for some* $B \in B_G$.

Theorem 2.27. $D^o_{sp}(G;Z) \neq \phi$ *if and only if one of the following two conditions is satisfied.*

(i) $\Delta > k_{sp}$. *item*(ii) $\Delta = k_{sp}$ *and there exists a vertex* u *of degree* Δ *such that* $N(u)$ *is not contained in a single block.*

Theorem 2.28. $D^o_{sp}(G;X_1) \neq \phi$ *if and only if* $\Delta \leq k_{sp}$.

Definition 2.29. *If* $A \subset V(G)$*, then* $N(A)$ *= the set of all neighbours of vertices in* A *and* $N[A] = A \cup N(A)$.

Definition 2.30. *For a complete block* B*,* $B^{+\Delta}$ *is obtained by the adjunction of one vertex each at every vertex of the block such that degrees of the adjoined vertices in the resulting graph are* Δ.

Example 2.31.

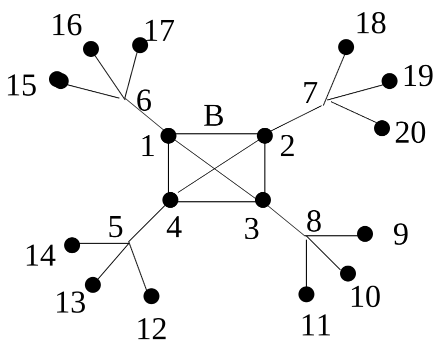

Figure 2.

$B^{+\Delta} = \langle\{1,2,3,4,5,6,7,8\}\rangle$.

$D = \{5,6,7,8,9,10,11,12,13,14,15,16,18,19,20\}$.

$V - D = \{1, 2, 3, 4\}$.
$V - D = B$, $B \cap D = \phi$, $V - D = B = P(B, D)$.

Theorem 2.32. *$D^o_{sp}(G; Y) \neq \phi$ if and only if*

(i) $\Delta \geq k_{sp}$ and

(ii) G has a block B which is a clique of order Δ and $\langle [N(B)] \rangle = B^{+\Delta}$.

Theorem 2.33. *$D^o_{sp}(G; Y) \neq \phi$ implies $\Delta \leq k_{sp} + 1$.*

Remark 2.34. *If $D^o_{sp}(G; Y) \neq \phi$, then $k_{sp} \leq \Delta \leq k_{sp} + 1$.*

Theorem 2.35. *$D^o_{sp}(G; X_2) \neq \phi$ if and only if the following condition are satisfied.*

(i) $\Delta \geq k_{sp}$.

(ii) V can be partitioned into four non empty sets V_1, V_2, V_3 and V_4 such that

(a) $V_1 \neq \phi$.

(b) $|V_1 \cup V_2| = \Delta$.

(c) $V_1 \cup V_2 \cup V_3 = B$ for some $B \in B_G$.

(d) $V_3 \in D_{sp}(V_2 \cup V_3)$.

(e) $\langle V_1 \rangle$ is complete, for every $x \in V_1$, $N(x) \cap V_2 = V_2$, $N(x) \cap V_3 = \phi$ and there exists $u \in N(x) \cap V_4$ with $d(u) = \Delta$.

Theorem 2.36. *If $\Delta > k_{sp}$, then the following statements are valid.*

(a) $N(u)$ is not contained in B for any $B \in B_G$ for any vertex u of degree Δ.

(b) $V - N(u) \in D^0_{sp}(G; Z)$ for any vertex u of degree Δ.

(c) $|N(u) \cap B| \leq \Delta - 1$ for any $B \in B_G$ and for any vertex u of degree Δ.

(d) $D^0_{sp}(G; X_1) = \phi$.

(e) $D^0_{sp}(G; Z) \neq \phi$.

Theorem 2.37. *$D^o_{sp}(G; X_2) \neq \phi$ implies $\Delta \leq k_{sp} + 1$.*

Observation 2.38. *$\gamma_{sp}(G) = n - \Delta$ if and only if $\Delta \geq k_{sp}$.*

Observation 2.39. $\gamma_s p(G) = n - k_{sp}$ *if and only if* $\Delta \leq k_{sp}$.

Theorem 2.40. $\gamma_p(G) = \gamma_{sp}(G)$ *if and only if one of the following three conditions is satisfied.*

(i) $\Delta > k$.

(ii) $\Delta = k$ *and* G *has a cut vertex with* $d(u) = \Delta$.

(iii) $\Delta \leq k$ *and* $k = k_{sp}$.

Proof. Let $\gamma_p(G) = \gamma_{sp}(G)$. If $\gamma_p(G) = n-\Delta$, then $\gamma_{sp}(G) = n-\Delta$. Then one of the two conditions (i) (or) (ii) is satisfied. If $\gamma_p(G) = n - k$, then $\gamma_{sp}(G) = n - k$.
Case (i) $\gamma_{sp}(G) = n - \Delta$.
$\gamma_p(G) = n - k = n - \Delta = \gamma_{sp}(G)$, then $\Delta = k$ and has a cut vertex u with $d(u) = \Delta$.
Case (ii) $\gamma_{sp}(G) = n - k_{sp}$.
Then $\gamma_p(G) = n - k$ and $\gamma_p(G) = \gamma_{sp}(G)$ implies $n - k = \gamma_p(G) = n - k_{sp} = \gamma_{sp}(G)$. Therefore, $k = k_{sp}$ and $\gamma_p(G) = n - k$ implies $n - k \leq n - \Delta$. That is, $k \geq \Delta$.
Conversely, If **(i)** is satisfied, then $n - \Delta < n - k$. Hence $\gamma_p(G) = n - \Delta$. $n-\Delta = \gamma_p(G) \leq \gamma_{sp}(G)$. Therefore, $\gamma_{sp}(G) = n-\Delta$. Therefore, $\gamma_p(G) = \gamma_{sp}(G)$.
If **(ii)** is satisfied, then $\gamma_p(G) = n - \Delta$ and hence $\gamma_{sp}(G) = n - \Delta$. That is, $\gamma_p(G) = \gamma_{sp}(G)$.
If **(iii)** is satisfied, then $\Delta \geq k$ implies $n - \Delta \leq n - k$ and hence $\gamma_p(G) = n - k$. Therefore, $\gamma_p(G) = n - k = n - k_{sp}$. Hence $n - k_{sp} = \gamma_p(G) \leq \gamma_{sp}(G)$. That is, $\gamma_{sp}(G) = n - k_{sp}$. Therefore, $\gamma_p(G) = \gamma_{sp}(G)$.

∎

Acknowledgement

We thank the referee for the useful comments and suggestions.

References

[1] B.D.Acharya, Purnima Gupta, *On point - set domination in graphs*, in; Proc. Nat. Seminar on Recent Development in Mathematics, N.M. Bujurke (Ed.), Karnataka University Press, Dharwad, 1996,106-108.

[2] B.D. Acharya, Purnima Gupta, *On point set domination in graphs: IV Separable graphs with unique minimum psd-sets*, Discrete Mathematics 195 (1999), 1 - 13.

[3] J.A. Bondy, U.S.R. Murty, *Graph Theory and Applications*, New York, North Holland, 1986.

[4] F.Harary, *Graph theory*, Addison - Wesley,1969.

[5] T.Haynes, S.T.Hedetniemi, P.J.Slater, *Fundamentals of domination in graphs*, Marcel Dekker Inc, New York, 1998.

[6] T.Haynes, S.T.Hedetniemi, P.J.Slater, *Advances in the theory of domination in graphs*, Marcel Dekker Inc, New York, 1998.

[7] E. Sampathkumar, L. Pushpalatha, *Point set domination number of a graph*, Indian J. Pure Appl. Math., 24(4)(1993), 225 - 229.

[8] E.Sampathkumar, *Graph Theory Research Report No:3*, Karnatak University, Dharwad, 1972.

[9] E. Sampathkumar, L. Pushpalatha *Strong weak domination and domination balance in a graph*, Discrete Math . 161 (1996), 235-242.

[10] H.B.Walikar, B.D.Acharya, E.Sampathkumar, *Recent advances in the theory of domination in graphs and its applications*, MRI Lecture Notes in Mathematics, Allahabad, No. 1, 1979.