*International Journal of Mathematics and Soft Computing*
*Vol.1, No.1 (2011), 1 - 8*

**ISSN 2249 - 3328**

# The superior complement in graphs

**KM. Kathiresan**
Center for Research and Post Graduate Studies in Mathematics
Ayya Nadar Janaki Ammal College
Sivakasi - 626 124,Tamil Nadu, India
E-mail: kathir2esan@yahoo.com

**G. Marimuthu** [1]
Department of Mathematics
The Madura College, Madurai - 626 011
Tamil Nadu, India
E-mail: yellowmuthu@yahoo.com

**C. Parameswaran**
Center for Research and Post Graduate Studies in Mathematics
Ayya Nadar Janaki Ammal College
Sivakasi - 626 124,Tamil Nadu, India
E-mail: parames65_c@yahoo.com

### Abstract

For distinct vertices $u$ and $v$ of a nontrivial connected graph $G$, we let $D_{u,v} = N[u] \cup N[v]$. We define a $D_{u,v}$-walk as a $u$-$v$ walk in $G$ that contains every vertex of $D_{u,v}$. The superior distance $d_D(u,v)$ from $u$ to $v$ is the length of a shortest $D_{u,v}$-walk. For each vertex $u \in V(G)$, define $d_D^-(u) = min\{d_D(u,v) : v \in V(G) - \{u\}\}$. A vertex $v(\neq u)$ is called a *superior neighbor* of $u$ if $d_D(u,v) = d_D^-(u)$. In this paper we define the concept of superior complement of a graph $G$ as follows: The superior complement of a graph $G$ is denoted by $\overline{G}_D$ whose vertex set is as in $G$. For a vertex $u$, let $A_u = \{v \in V(G) : d_D(u,v) \geq d_D^-(u) + 1\}$. Then $u$ is adjacent to all the vertices $v \in A_u$ in $\overline{G}_D$. The main focus of this paper is to prove that there is no relationship between the superior diameter $d_D(G)$ of a graph $G$ and the superior diameter $d_D(\overline{G}_D)$ of the superior complement $\overline{G}_D$ of $G$.

**Key words**: Superior distance, superior radius, superior diameter,superior neighbor, superior complement.

**AMS Subject Classification(2010):** 05C12.

[1]Research supported by UGC, Government of India, MRP-2794/09/09(MRP/UGC-SERO).

## 1 Introduction

By a graph we mean a nontrivial undirected graph without loops and multiple edges. As usual $V(G)$ denotes the set of vertices of a graph $G$ and $E(G)$ denotes the set of edges of $G$. The distance between the vertices $u$ and $v$ is the length of a shortest $u$-$v$ path in $G$. The distance to a vertex farthest from a vertex $u$ is the eccentricity $e(u)$ of the vertex $u$ in $G$. The minimum among the eccentricities is called the radius $r(G)$ of $G$ and the maximum among the eccentricities is called the diameter $d(G)$ of $G$. By a neighbor of a vertex $u$, we mean, any vertex whose distance from $u$ is minimum. If $v$ is a vertex distinct from $u$ whose distance from $u$ is minimum, then this distance $d(u, v)$ must be 1. For each vertex $u \in V(G)$, define $d^-(u) = min\{d(u, v) : v \in V(G) - \{u\}\}$. A vertex $v \neq u$ is called a neighbor of $u$ if $d^-(u) = d(u, v)$. Since $d^-(u) = 1$ for all $u \in V(G)$, this is equivalent to the standard definition of neighbor. For graph theoretic notation and terminology, we follow [1].

If $X$ and $Y$ are two cities, then for a taxi driver, the distance between $X$ and $Y$ is the actual distance between the two cities. However for a driver of a passenger bus, the distance between the same two cities is greater than the usual distance since he has to visit important places in and around the two cities to pickup and drop off passengers. So a driver of a passenger bus has to find a shortest route that begins at $X$, ends at $Y$, and passes through each of the neighboring places of $X$ and $Y$.

Kathiresan and Marimuthu [2] defined a variation of distance that models the bus route just described. For two vertices $u$ and $v$ of $G$, let $D_{u,v} = N[u] \cup N[v]$. They define a $D_{u,v}$-walk as a $u$-$v$ walk in $G$ that contains every vertex of $D_{u,v}$. The superior distance $d_D(u, v)$ from $u$ to $v$ is the length of a shortest $D_{u,v}$-walk. If $u$ and $v$ are in the different components of a disconnected graph, then we define $d_D(u, v) = \infty$.

The superior eccentricity $e_D(u)$ of a vertex $u$ is the superior distance to a vertex farthest from $u$. A vertex $v$ is called a superior eccentric vertex of a vertex $u$ if $e_D(u) = d_D(u, v)$. The minimum among the superior eccentricities in a graph $G$ is called the superior radius $r_D(G)$ of $G$ and the maximum among the superior eccentricities in a graph $G$ is called the superior diameter $d_D(G)$ of $G$.

One variation of domination is seen in [3] where the concept of superior domination in graphs is defined. In the standard definition of domination in a graph, a vertex $u$ dominates itself and each of its neighbors. For each vertex $u \in V(G)$, define $d_D^-(u) = min\{d_D(u,v) : v \in V(G) - \{u\}\}$. A vertex $v(\neq u)$ is called a superior neighbor of $u$ if $d_D(u,v) = d_D^-(u)$. The superior neighborhood $N_D(u)$ of a vertex $u$ is the set of all superior neighbors of $u$; and its closed superior neighborhood is $N_D[u] = N_D(u) \cup \{u\}$. A vertex $u$ is said to superior dominate a vertex $v$ if $v$ is a superior neighbor of $u$. A set $S$ of vertices of $G$ is called a superior dominating set if every vertex of $V - S$ is superior dominated by some vertex of $S$. The complement $\overline{G}$ of a graph $G$ is the graph with vertex set as in $G$ and two vertices $u$ and $v$ are adjacent in $\overline{G}$ if and only if $d(u,v) \geq 2$. As there is a complement $\overline{G}$ of a graph $G$ with respect to usual distance, it is natural to define a new complement of a graph with respect to the superior distance. There are some cases in which the usual complement and the superior complement coincide. For example $\overline{C_4} = \overline{(C_4)}_D$.

In this paper, we define the concept of superior complement in graphs as follows: The superior complement of a graph $G$ is denoted by $\overline{G}_D$ whose vertex set is as in $G$. For each vertex $u \in V(G)$, let $A_u = \{v \in V(G) : d_D(u,v) \geq d_D^-(u)+1\}$. Then $u$ is adjacent to all the vertices $v$ in $A_u$ in $\overline{G}_D$.

Next we provide one result which is found in [4].

**Proposition A.**[4] *Let $G$ be a nontrivial connected graph with $N[u] \cup N[v] = V(G)$ for any two distinct vertices $u$ and $v$. Then $d_D(u,v) = n-1$ for any two distinct vertices $u$ and $v$ if and only if $G$ is hamiltonian connected.*

## 2 The Superior Complement

**Observation 2.1.** *For any connected graph $G$ of order $n$, $d_D(G) \leq 2n-3$.*

The sharpness of the upper bound is satisfied by the graphs stars and the double stars.

**Theorem 2.2.** *Let $G$ be a graph of order $n$. Then $\overline{G}_D = K_n$ if and only if $G$ is totally disconnected.*

**Proof.** Assume that $\overline{G}_D = K_n$. We claim that $G = \overline{K}_n$. Suppose $G$ has at least one edge $uv$ (say). Then by the definition of $\overline{G}_D, uv \notin \overline{G}_D$, a contradiction.

If $G = \overline{K}_n$, then $\overline{G}_D = K_n$ is followed from the definition. ∎

**Proposition 2.3.** *If $G = K_{1,n-1}$, then $\overline{G}_D = G$.*

**Observation 2.4.** *Let $G$ be a graph of order $n$. Then $\overline{G}_D = \overline{G}$ if and only if $N(u) = N_D(u)$ for all $u \in V(G)$.*

**Theorem 2.5.** *Let $G$ be a nontrivial connected graph of order $n$ such that $N[u] \cup N[v] = V(G)$ for each pair of distinct vertices $u$ and $v$. Then $G$ is hamiltonian connected if and only if $\overline{G}_D = \overline{K_n}$.*

**Proof.** Assume that $G$ is hamiltonian connected. Then by Proposition A, $d_D(u, v) = n - 1$ for each pair of distinct vertices $u$ and $v$. Thus for a vertex $u$, every vertex except $u$ is a superior neighbor of $u$ and hence $\overline{G}_D = \overline{K}_n$.

Conversely assume that $\overline{G}_D = \overline{K}_n$. It is enough if we prove that $d_D(u, v) = n - 1$ for each pair of distinct vertices $u$ and $v$. Since $\overline{G}_D = \overline{K}_n$, for each vertex $u$, all the vertices except $u$ are the superior neighbor of $u$ in $G$. Thus $d_D(u, v) = l$ for all $u, v \in V(G)$ in $G$. We claim that $n - 1 = l$.

Suppose there exists a pair of vertices $u$ and $v$ such that $d_D(u, v) = l < n - 1$. Then $v$ is a superior neighbor of $u$ and there are some vertices in $A_u$. Thus there is at least one edge in $\overline{G}_D$, a contradiction.

Suppose there exist a pair of vertices $u$ and $v$ such that $d_D(u, v) = l > n - 1$. Then $u$ and $v$ are non superior neighbor to each other. Thus $uv \in E(\overline{G}_D)$. This is a contradiction. ∎

**Lemma 2.6.** *Let $G$ be a hamiltonian connected graph of order $n \geq 2$ such that $N[u] \cup N[v] = V(G)$ for each pair of distinct vertices $u$ and $v$. Then $d_D(\overline{G}_D) = \infty$.*

**Proof.** Let $G$ be a Hamiltonian connected graph of order $n \geq 2$ such that $N[u] \cup N[v] = V(G)$ for each pair of vertices $u$ and $v$. Then by Proposition A, $d_D(u, v) = n - 1$, for each pair of vertices $u$ and $v$. Then $d_D(G) = r_D(G) = n - 1$.

It follows that $d_D^-(u) = n - 1$ for all $u \in V(G)$. Thus for each vertex $u$, every vertex of $G$ other than $u$ is a superior neighbor of $u$. Thus $\overline{G}_D = \overline{K}_n$ and hence $d_D(\overline{G}_D) = \infty$. ∎

**Proposition 2.7.** *For every $n \geq 5$, there exists a connected graph $G$ of order $n$ such that $\overline{G}_D = \overline{G}$.*

**Proof.** Let $G$ be any cycle $C_n, n \geq 5$ on $n$ vertices. Let $u$ be any vertex of $C_n$.

Then $d_D(u,v) = 5$ if $v$ is adjacent to $u$ and $d_D(u,v) > 5$ if $v$ is nonadjacent to $u$ in $C_n$.
Thus $N_D(u) = N(u)$ for all $u \in V(C_n)$. By Observation 2.4, $\overline{G}_D = \overline{G}$. ∎

**Remark 2.8.** *If $G = K_n$, then $\overline{G}_D = \overline{G}$ for any $n$.*

**Remark 2.9.** *Also $\overline{G}_D$ is a hamiltonian connected graph such that $N[u] \cup N[v] = V(\overline{G}_D)$ for each pair of vertices $u$ and $v$. By Proposition A, $d_D(u,v) = n-1$ for any $u$ and $v$.*

**Corollary 2.10.** *There exists a graph $G$ such that $\overline{H}_D \neq G$ where $H = \overline{G}_D$.*

The following results are found in [1].

**Theorem 2.11.** *If $G$ is a simple graph with diameter at least 3, then $\overline{G}$ has diameter at most 3.*

**Theorem 2.12.** *If $G$ is a simple graph with diameter at least 4, then $\overline{G}$ has diameter at most 2.*

Even though there is a relation between the diameter of $G$ and the diameter of $\overline{G}$ in the usual distance, we show that there is no such relationship between the superior diameter $d_D(G)$ of a graph $G$ and the superior diameter of the superior complement $\overline{G}_D$ of $G$.

**Lemma 2.13.** *For each positive integer $n \geq 4$, there exists a connected graph $G$ of order $n$ such that $d_D(\overline{G}_D) = 2n-4$ where $d_D(G) = n$.*

**Proof.** Consider the graph $K_{n-1}$ for any $n \geq 4$. Let $u$ be the new vertex and let $u_i, i = 1,2,\ldots,n-1$ be the vertices of $K_{n-1}$. Now join $u$ with any one of the vertices of $K_{n-1}$. Assume that $u$ is adjacent to $u_1$. Let the resulting graph be $G$. Then $d_D(u,u_1) = n$ in $G$. The superior eccentric vertex of each vertex of $G$ except $u_1$ is $u_1$. Thus $e_D(u) = n$ for all $u$ in $G$. $d_D(u,u_i) = n-1$ for all $i = 2,3,\ldots,n-1$; $d_D(u_i,u_j) = n-2$ for all $i \neq j, i,j = 2,3,\ldots,n-1$. Thus each $u_i, i = 2,3,\ldots,n-1$ is adjacent to $u$ and $u_1$ in $\overline{G}_D$. Also $u$ is adjacent to $u_1$ in $\overline{G}_D$.

Now, $d_D(u, u_1) = 2n - 4$ and $d_D(u_1, u) = 2n - 4$ in $\overline{G}_D$. $d_D(u_i, u) = 2n - 5$ for all $u_i, i = 2, 3, \ldots, n-1$; $d_D(u_i, u_j) = 4$ for all $i \neq j, i, j = 2, 3, \ldots, n-1$ in $\overline{G}_D$. Thus $d_D(\overline{G}_D) = 2n - 4$. ∎

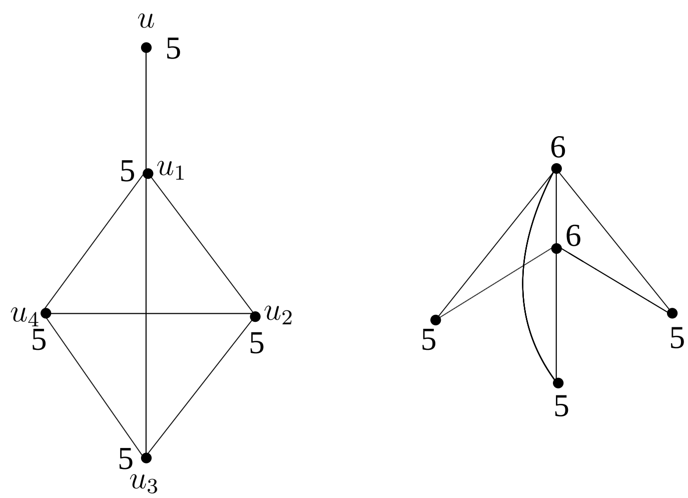


Figure 1: The graphs $G$ and $\overline{G}_D$ with superior eccentricity with five vertices

**Remark 2.14.** *The direct verification shows that the result is not true for* $n \leq 3$.

**Lemma 2.15.** *Let* $G$ *be a complete bipartite graph* $K_{m,n}$. *Then the followings hold.*

(a) $\overline{G}_D = \begin{cases} K_m \cup K_n & \text{if } m = n \\ K_m + \overline{K_n} & \text{if } m < n \end{cases}$

(b) $d_D(G) = d_D(\overline{G}_D) = 2n$ *if* $n > m = 2$.

**Proof.** (a) Let $V_1$ and $V_2$ be the bipartition of the vertex set of $G$ such that $|V_1| = m$ and $|V_2| = n$. Assume that $m = n$. Let $u$ and $v$ be the vertices in the same partite set. Then $d_D(u, v) = 2m$. If $u$ and $v$ belong to different sets, then $d_D(u, v) = 2m - 1$. Thus for each vertex $u \in V_1$, every vertex in $V_2$ is a superior neighbor of $u$. Thus $u$ is adjacent to all other vertices in $V_1$ in $\overline{G}_D$. Similarly each vertex $u$ in $V_2$ is adjacent to all other vertices in $V_2$ in $\overline{G}_D$. Hence $\overline{G}_D = K_m \cup K_n$, a disconnected graph.

Now, assume that $m < n$. Then every vertex of $V_1$ is a superior eccentric vertex of all other vertices in $V_1$. Let $u \in V_1$ and $v \in V_2$. Then $d_D(u, v) = 2n - 1$; $d_D(u, v) = 2n$ if $u, v \in V_1$; $d_D(u, v) = 2m$ if $u, v \in V_2$ in $G$. Thus $\overline{G}_D = K_m + \overline{K_n}$.

(b) Assume that $n > m = 2$. Then $d_D(u,v) = 2n$ if $u, v \in V_1$; $d_D(u,v) = 2n-1$ if $u \in V_1$ and $v \in V_2$; $d_D(u,v) = 2m$ if $u, v \in V_2$ in $G$. This is also true in $\overline{G}_D$, since $\overline{G}_D = K_2 + \overline{K_n}$, a connected graph. ∎

**Lemma 2.16.** *For each odd $l$, there exists a connected graph $G$ such that $d_D(G) = d_D(\overline{G}_D) = l$.*

**Proof.** Consider the graph $K_{m,n}$. Assume that $m = 1$ and $n \geq 1$. When $n = 1$ the result is trivial. For otherwise, let $u$ be the vertex of degree $n$ in $G$. Then $d_D(u,v) = 2n-1$ for all $v \in V_2$; $d_D(w,v) = 2$ if $w, v \in V_2$. Also $d_D(v,u) = 2n-1$ for all $v \in V_2$. Clearly, $G = \overline{G}_D$. Thus $d_D(G) = d_D(\overline{G}_D) = 2n-1 = l$ (say). ∎

**Proposition 2.17.** *There exist distinct graphs $G$ and $H$ with $\overline{G}_D \cong \overline{H}_D$.*

**Proof.** Let $G$ be the graph constructed in Lemma 2.13. Take $H = K_{2,n-2}$. Then $\overline{G}_D = \overline{H}_D$. ∎

**Lemma 2.18.** *There exist graphs $G$ such that $d_D(\overline{G}_D) < d_D(G)$.*

**Proof.** Let $C_m$ be a cycle, $m = 3, 4, 5, \ldots, 8$. Attach a path of length 2 at any vertex of the cycle $C_m$ and let the resulting graph be $G_m$. It is easy to verify that $d_D(G_m) \leq 10$; $d_D\overline{(G_m)}_D \leq 9$ and $d_D\overline{(G_m)}_D < d_D(G)$ for all $G_m$. ∎

**Theorem 2.19.** *There is no relationship between the superior diameter $d_D(G)$ of a graph $G$ and the superior diameter of the superior complement $\overline{G}_D$ of $G$.*

**Proof.** Follows from Lemmas 2.13, 2.15, 2.16 and 2.17. ∎

**Open Problems:**

1. Characterize all graphs $G$ for which $\overline{G}_D = G$.

2. Characterize all graphs $G$ for which $\overline{H}_D = G$ where $H = \overline{G}_D$.

3. Given any natural numbers $a$ and $b$ (large enough), does there exist a graph $G$ such that $d_D(G) = a$ and $d_D(\overline{G}_D) = b$?

## References

[1] F. Buckley, F. Harary, *Distance in Graphs*, Addison-Wesley Reading, 1990.

[2] KM. Kathiresan, G. Marimuthu, *Superior distance in graphs*, J. Combin. Math. Combin.Comput., 61(2007), 73-80.

[3] KM. Kathiresan, G. Marimuthu, *Superior domination in graphs*, Util. Math., 76(2008), 173-182.

[4] G. Marimuthu, *A study on the radius and diameter of graphs*, Ph.D Thesis, Madurai Kamaraj University, Madurai, India, 2006.