Transient Analysis of Hysteresis Queueing Model Using Matrix Geometric Method

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ABSTRACT

Various analytical methods have been proposed for the transient analysis of a queueing system in the scalar domain. In this paper, a vector domain based transient analysis is proposed for the hysteresis queueing system with internal thresholds for the efficient and numerically stable analysis. In this system arrival rate of customer is controlled through the internal thresholds and the system is analyzed as a quasi-birth and death process through matrix geometric method with the combination of vector form Runge-Kutta numerical procedure which utilizes the special matrices. An arrival and service process of the system follows a Markovian distribution. We analyze the mean number of customers in the system when the system is in transient state against varying time for a Markovian distribution. The results show that the effect of oscillation/hysteresis depends on the difference between the two internal threshold values.

Key Words: **Hysteresis, Runge-Kutta, Threshold, Transient, Markov Distribution.**

1. INTRODUCTION

The queueing theory is the efficient way to analyze any analog/digital communication system. The performance of any analog / digital communication system can be evaluated in either continuous or discrete domain by modeling the system through queueing system [1]. The analysis of continuous time queueing system in which arrival rate of customers have major impact to control the overloading or congestion in the system can be efficiently done in vector domain as compared to the scalar domain analysis [2].

Majorly the analysis of continuous time queueing system is carried out in scalar domain by obtaining the Kolmogorov differential equations for all possible system states [3]. The manipulation of Kolmogorov differential equations to obtain the performance measures is very critical as the system states increases and some time it becomes impossible to find the close form solution. To avoid or overcome this difficulty, a vector domain analysis is efficient to solve such systems in which the system states grows towards infinity [4]. A vector domain based

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numerical method called matrix geometric method can be efficiently used either system states are finite or infinite [5-6].

The steady state analysis needs that the system must be stable or run for a long period. But it is not always possible that the system can run for a longer period. Accordingly, a transient analysis is most important to analyze the system behavior as the time changes [7]. The transient analysis of such system in scalar domain is more complicated because it involves the differential equations and the scalar form of Runge-Kutta numerical procedure. The matrix geometric method is efficient method to analyze the system behavior in transient mode by developing the Runge-Kutta numerical procedure in vector form [8]. In this method, a whole system is converted into a generator matrix then the system is partitioned into boundary and repetition portions to obtain the sub-matrices of the system [9]. These sub-matrices are the input to the Runge-Kutta numerical procedure for the transient analysis. This method is much more efficient and requires less computation as compare to the scalar method [10].

Tadi, L., et. al. [11] studied the performance of hysteretic system in which an r-quorum queueing system with random server capacity under N-policy discipline was considered in scalar domain. In [12] a non-preemptive delay priority queueing system with a hysteresis mechanism for priority control in ATM networks was analyzed. In [13] a threshold-based multi-server queuing model with hysteresis is analyzed to evaluate the behavior of the voice on demand system in the scalar form.

The hysteresis queueing model with two internal thresholds and controlled arrival rate [14] have been solved as a quasi birth and death process using matrix geometric method along with vector form Runge-Kutta numerical procedure. This system has more than two boundaries and repeating structures in a structured Markov chain and system is analyzed for their transient behavior.

The rest of the paper is organized as follows. Markovian distribution, matrix geometric method and hysteresis queueing model are discussed in Sections 2. In Section 3 matrix geometric analysis of hysteresis model with internal thresholds and Markovian distribution is discussed. The Sections 4 and 5 present the results of transient analysis and conclusion respectively.

2. MODELS AND DEFINATIONS

In this section, we first describe the Markovian distribution followed by matrix geometric method. Next, we present hysteresis queueing model.

2.1 Markovian Distribution

A Markovian process or probability distribution is defined as the probability distribution of the future states only depends upon the present states not on the past states. The Markovian distribution exhibits the memoryless property.

2.2 Matrix Geometric Method

MGM (Matrix Geometric Method) is the analytical method to solve the structured Markov chain in vector form. It utilizes the special feature of the structured Markov chain which is the repetition of the states [14]. This method consists of different number of steps to solve the structured Markov chain. These steps are:

(i) Develop a queueing model.

(ii) Construct a structured Markov chain.

(iii) Develop a generator matrix in lexicographical order.

(iv) Partitioning the structured Markov chain in boundary and repetition portions.

(v) Obtaining sub-matrices of boundary and repetition portions.

(vi) Compute the rate matrix R.

vii) Compute the initial probability vectors π_0 and π_1.

(viii) Compute the performance measures through rate matrix and initial probability vectors.

2.3 Hysteresis Queueing Model with Internal Thresholds and Markovian Distribution

The hysteresis queueing model with two internal thresholds and Markovian distribution is shown in Fig. 1. This system handles the finite capacity of customers with queue capacity S. The customers arrive in the system with two different arrival rates which follows the Markovian distribution. These arrival rates are categorized as normal λ_n and reduced λ_r rates. The arrival rates of the system are controlled through the two internal thresholds (S_{go}, S_{stop})of the system. The customers are served through one server in which service process is Markovian distribution with rate μ.

Initially the customers arrive in the queue with the normal rate λ_n up to internal threshold S_{stop}. Once the customers in the queue reached at second internal threshold S_{stop} the arrival rate of customers will be switched from normal rate λ_n to the reduced rate λ_r upto the maximum capacity of the queue S. The arrival rate will be switched back from reduced rate λ_r to the normal rate λ_n, when the number of customers in the queue will reached to the first internal threshold S_{go}. The arrival rate is controlled through the two internal thresholds to avoid the system full. The customer service rate follows a Markovian distribution with rate μ.

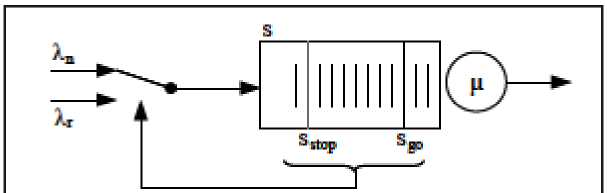

FIG. 1. HYSTERESIS QUEUEING MODEL WITH INTERNAL THRESHOLDS AND MARKOVIAN DISTRIBUTION

3. MATRIX GEOMETRIC ANALYSIS OF HYSTERESIS QUEUEING MODEL WITH INTERNAL THRESHOLDS AND MARKOVIAN DISTRIBUTION

The structured Markov chain of the hysteresis queueing model with two internal thresholds is constructed as shown in Fig. 2. The structured Markov chain of the system becomes a quasi-birth death process and system state space of the system is represented by the (number of customers in the system, i.e. arrival rate). When the arrival rate is normal λ_n, then the system states are represented as (0/n), (1/n), …, (S_{stop}-1/n) up to second internal threshold. When the second threshold reached, the system states becomes (S_{stop}/r), (S_{stop}+1/r), … (S/r). The system states (S_{stop}/r), (S_{stop}-1/r), … (S_{go}/r) represented the condition when system switched the arrival rate from reduced to normal by reaching towards the first internal threshold. The system states (S_{stop}-1/r) and (S_{stop}/r) represents the situation when normal arrival rate switched to the reduced rate and similarly (S_{go}+1/r) and (S_{go}/n) system states represents the switching between reduced and normal arrival rate. The Fig. 2 shows the structured Markov chain for the maximum capacity 12, first and second internal thresholds are 4 and 9 respectively.

The finitesimal generator matrix from the structured Markov chain is represented by Equation (1) in which system state space is arranged in lexicographical order:

$$Q=\begin{array}{c|cccccccccccccccc}
 & 0 & 1 & 2 & 3 & 5n & 5r & 6n & 6r & 7n & 7r & 8n & 8r & 9 & 10 & 11 & 12 \\ \hline
0 & -\lambda_n & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \mu & a & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & \mu & a & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & \mu & a & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & \mu & a & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5n & 0 & 0 & 0 & 0 & \mu & a & 0 & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5r & 0 & 0 & 0 & 0 & \mu & 0 & b & \lambda_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6n & 0 & 0 & 0 & 0 & 0 & \mu & 0 & a & \lambda_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6r & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & b & 0 & \lambda_n & 0 & 0 & 0 & 0 & 0 \\
7n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & a & 0 & \lambda_n & 0 & 0 & 0 & 0 \\
7r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & b & 0 & \lambda_r & 0 & 0 & 0 \\
8n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & b & \lambda_n & 0 & 0 & 0 \\
8r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & b & \lambda_r & 0 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & b & \lambda_r & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & b & \lambda_r & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & b & \lambda_r \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -\mu
\end{array} \qquad (1)$$

where $\alpha = -(\lambda_n + \mu)$ $b = -(\lambda_r + \mu)$.

To simplify the analysis of the model, we will rearrange the structured Markov chain into three subsystems according to the arrival rate of the customers as shown in Fig. 3. The first subsystem structured Markov chain is consider from when system is idle to the first internal threshold S_{go}, where only normal arrival rate λ_n is allowed. Where as the structured Markov chain of second subsystem is considered between two internal thresholds S_{go} and S_{stop}, where customers are permitted with both arrival rates (λ_n, λ_r) depends on the thresholds. Finally the third subsystem is consider from the second internal threshold S_{stop} to the maximum capacity of queue S, where customers are only allowed with reduced arrival rate λ_r.

According to the simplification of the structured Markov chain, the finitesimal generator matrix is partitioned into boundary and repeating system states of each subsystem to obtain the sub- matrices of each subsystem (Equation(2)).

$$Q=\begin{pmatrix}
-\lambda_n & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\mu & a & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \mu & a & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \mu & a & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \mu & a & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mu & a & 0 & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mu & 0 & b & 0 & \lambda_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \mu & 0 & a & 0 & \lambda_n & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & b & 0 & \lambda_r & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & a & 0 & \lambda_n & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & b & 0 & \lambda_r & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & a & 0 & \lambda_n & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & b & \lambda_r & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & b & \lambda_r & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & b & \lambda_r & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & b & \lambda_r\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & -\mu
\end{pmatrix} \quad (2)$$

The sub-matrices of first subsystem are shown in Equation (3), where A_0, A_1, A_2 are sub-matrices of repeating portion. B_0, B_1, B_2 are initial boundary sub-matrices and B_3 is the final boundary sub-matrix.

$$B_0 = [-\lambda_n],\ B_1 = A_0 = -[\lambda_n], A_1 = [-(\lambda_n + \mu)],\ B_2 = A_2 = [\mu] \quad (3)$$

The sub-matrices of second subsystem are shown in Equation (4). The initial and final

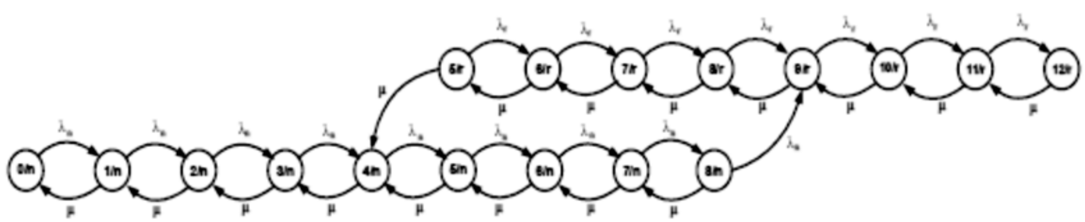

FIG. 2. STRUCTURED MARKOV CHAIN: S_{go}=4, S_{stop}=9 AND S=12

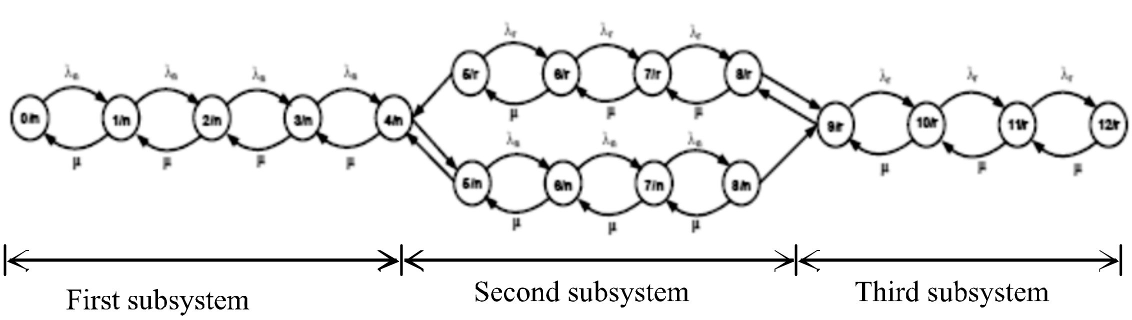

FIG. 3. SIMPLIFIED STRUCTURED MARKOV CHAIN

$$B_3 = \left[-\lambda_n + \mu\right], B_4 = \left[\lambda_n 0\right], B_2 = \begin{pmatrix} \mu \\ \mu \end{pmatrix}, \bar{A}_1 = \begin{pmatrix} (-\lambda_n + \mu) & 0 \\ 0 & (-\lambda_r + \mu) \end{pmatrix}$$
$$\bar{A}_0 = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_r \end{pmatrix}, \bar{A}_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \quad (4)$$

boundary sub-matrices are represented by B_3, B_4, B_5 and B_6. Where $\bar{A}_0$, $\bar{A}_1$ and $\bar{A}_2$ are sub-matrices of repetition portion. The boundary and repeating sub-matrices of third subsystem are shown in Equation (5).

$$B_6 = \begin{pmatrix} (-\lambda_n + \mu) & 0 \\ 0 & (-\lambda_r + \mu) \end{pmatrix}, B_7 = \begin{pmatrix} \lambda_n \\ \lambda_r \end{pmatrix}, B_8 = \begin{pmatrix} 0 & \mu \end{pmatrix},$$
$$\hat{A}_1 = (-\lambda_r + \mu), \hat{A}_0 = (\lambda_r), \hat{A}_2 = (\mu) \quad (5)$$

Where the repeating and boundary portions sub-matrices are $\hat{A}_0$, $\hat{A}_1$, $\hat{A}_2$, B_6, B_7, B_8 and B_9. By substituting the sub-matrices in the finitesimal generator matrix then the finitesimal generator matrix becomes block finitesimal generator matrix as shown in Equation (6).

$$Q = \begin{array}{c|ccccccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
0 & B_0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & B_2 & A_1 & A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & A_2 & A_1 & A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & A_2 & A_1 & A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & A_2 & B_3 & B_A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & B_5 & \bar{A}_1 & \bar{\Lambda}_0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
6 & 0 & 0 & 0 & 0 & \bar{A}_2 & \bar{A}_1 & \bar{A}_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & \bar{A}_2 & \bar{A}_1 & \bar{A}_0 & 0 & 0 & 0 & 0 & 0 \\
8 & 0 & 0 & 0 & 0 & 0 & \bar{A}_2 & B_6 & B_7 & 0 & 0 & 0 & 0 & 0 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_8 & \hat{A}_1 & \hat{A}_0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{A}_2 & \hat{A}_1 & \hat{A}_0 & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{A}_2 & \hat{A}_1 & A_0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{A}_2 & B_9
\end{array} \quad (6)$$

The resulting structure of block finitesimal generator matrix (Equation (6)) is same as the structure of finite QBD process.

4. ANALYTICAL RESULTS

An analytical program of matrix geometric method and Runge-kutta procedure is written in visual C^{++} which utilizes the sub-matrices which are obtained through matrix geometric method.

The transient analysis of the hysteresis model with internal thresholds having Markovian distribution is shown in Fig. 4 for various system sizes, S_{go} and S_{stop} thresholds.

Fig. 4 shows the mean number of customers in the system when the system is in transient state against varying the time for the Markovian distribution. It is observed that the system length is zero up to time t = 10 and then there is a sudden increase in the system length. Initially mean system length increases to the threshold S_{stop} and then it falls to reduce the mean system length towards the S_{go} threshold. Here, it is clearly seen that there is again increase in the mean system length before reaching to the S_{go} threshold. The mean system length oscillates between the S_{go} and S_{stop} thresholds and system will not reach its maximum capacity.

It is also observed that the effect of oscillation depends on the difference between the S_{go} and S_{stop} thresholds. The graph clearly shows that the smaller difference between the S_{go} and S_{stop} thresholds have smaller oscillation and the system will be stabilized soon and the system with larger difference between the S_{go} and S_{sto} thresholds have more oscillations.

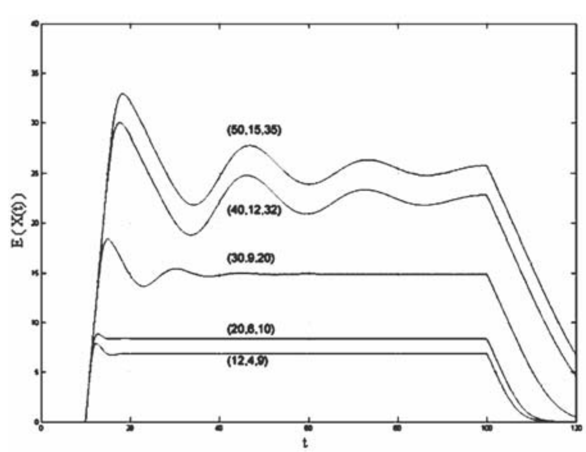

FIG. 4. MEAN NUMBER IN THE SYSTEM: TRANSIENT ANALYSIS WITH MARKOVIAN DISTRIBUTION λ_n=6, λ_r=01., μ=1, c_v=,6, (SYSTEM CAPACITY, S_{go}, S_{stop})

5. CONCLUSION

In this paper, we used MGM along with vector form Runge-Kutta numerical procedure to evaluate the transient behavior of the hysteresis queueing model with two internal thresholds. MGM utilizes the special features of the structured Markov chain to obtain the sub-matrices for the Runge-Kutta numerical procedure for the transient analysis.

A continuous time queueing analysis is used in contrast with MGM and Runge-Kutta procedure to evaluate the controlled arrival rate hysteresis queueing model with internal thresholds for the Markovian arrival and service processes. The results of mean number in the system by varying the values of the internal thresholds as well as capacity of queue in a transient state are presented. In terms of transient analysis of hysteresis queueing model with internal thresholds, MGM with Runge-Kutta vector form procedure can be used more effectively and quickly in solving the large systems which has more than two boundaries in structured Markov chain and requires solution of Kolmogrove differential equations and needs enormous numerical methods

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