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Schur-geometric and Schur-harmonic Convexity of an Integral Mean for Convex Functions

Jian Sun[1], Zhi-Ling Sun[1], Bo-Yan Xi[1], Feng Qi[2,3,*]

[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, China
[2]Department of Mathematics, School of Science, Tianjin Polytechnic University, Tianjin City, China;
[3]Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, Henan Province, China
*Corresponding author: qifeng618@gmail.com

Abstract In the paper, Schur-geometric and Schur-harmonic convexity of an integral mean for convex functions are established.

Keywords: *Schur-convex function; Schur-geometrically convex function; Schur-harmonically convex function; inequality; generalized logarithmic mean*

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1. Introduction

In [3], N. Elezović and J. Pečarić established the following theorem.

Theorem A ([3]). Let $f: I \subseteq R \to R$ and $a, b \in I^\circ$. Then

$$F(a,b) = \begin{cases} \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\mathrm{d}x, & a \neq b, \\ f(a), & a = b \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

In [7,10], Theorem A was generalized as the following theorem.

Theorem B ([7,10]). Let f be a continuous function and p a positive continuous weight on an interval I. Then the weighted arithmetic mean of f with weight p defined by

$$G(x,y) = \begin{cases} \dfrac{\int_x^y p(t)f(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t}, & x \neq y, \\ f(x), & x = y \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if

$$\frac{\int_x^y p(t)f(t)\mathrm{d}t}{\int_x^y p(t)\mathrm{d}t} \le \frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)}$$

holds (reverses) for all $(x,y) \in I^2$.

For more information on this topic, please refer to [5,8,9] and closely-related references therein.

In this paper, we discuss Schur-geometric and Schur-harmonic convexity of the mean $F(a,b)$ and obtain two results which generate Theorem A.

2. Definitions and Lemmas

In order to prove our main results we need the following definitions and lemmas.

Definition 1 ([4]). Let $I \subseteq R$ and $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in I^n$, and let $\varphi: I^n \to R$.

(1) x is said to be majorized by y (in symbols $x \prec y$) if $\sum_{i=1}^k x_{[i]} \le \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \cdots, n-1$ and $\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}$, where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of x and y in a descending order.

(2) $x \le y$ means $x_i \le y_i$ for all $i = 1, 2, \cdots, n$. ϕ is said to be increasing if $x \le y$ implies $\varphi(x) \le \varphi(y)$. ϕ is said to be decreasing if and only $-\phi$ is increasing.

(3) ϕ is said to be a Schur-convex function on I^n if $x \prec y$ on I^n implies $\varphi(x) \le \varphi(y)$. ϕ is said to be a Schur-concave function on I^n if and only $-\phi$ is Schur-convex function.

Definition 2 ([1,2]). Let $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in I^n \subseteq R_+^n$ and $\varphi: I^n \to R$ and let $\ln x = (\ln x_1, \cdots, \ln x_n)$, $\frac{1}{x} = \left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right)$.

(1) ϕ is said to be a Schur-geometrically convex function on I^n if $\ln x \prec \ln y$ on I^n implies $\varphi(x) \le \varphi(y)$. ϕ is said to be a Schur-geometrically concave function on

I^n if and only $-\phi$ is Schur-geometrically convex function.

(2) ϕ is said to be a Schur-harmonically convex function on I^n if $\frac{1}{x} \prec \frac{1}{y}$ on I^n implies $\varphi(x) \le \varphi(y)$. ϕ is said to be a Schur-harmonically concave function on I^n if and only $-\phi$ is Schur-harmonically convex function.

Lemma 2.1([1]). Let $\varphi: I^2 \subseteq R_+^2 \to R$ be a continuous function on I^2 and differentiable in interior of I^2. Then φ is Schur-geometrically convex (Schur-geometrically concave) if and only if it is symmetric and

$$(b-a)\left(b\frac{\partial\varphi}{\partial b} - a\frac{\partial\varphi}{\partial a}\right) \ge (\le) 0$$

for all $a, b \in I$.

Lemma 2.2 ([2]). Let $\varphi: I^2 \subseteq R_+^2 \to R$ be a continuous function on I^2 and differentiable in interior of I^2. Then φ is Schur-harmonically convex (Schur-harmonically concave) if and only if it is symmetric and

$$(b-a)\left(b^2\frac{\partial\varphi}{\partial b} - a^2\frac{\partial\varphi}{\partial a}\right) \ge (\le) 0$$

for all $a, b \in I$.

For two positive numbers $a > 0$ and $b > 0$, define

$$A(a,b) = \frac{a+b}{2},$$
$$G(a,b) = \sqrt{ab},$$
$$H(a,b) = \frac{2ab}{a+b}$$

and

$$L_r(a,b) = \begin{cases} \left(\dfrac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right)^{1/r}, & a \ne b, r \ge 1, \\ a, & a = b. \end{cases}$$

It is well known that $A(a,b)$, $G(a,b)$, $H(a,b)$ and $L_s(a,b)$ are respectively called the arithmetic, geometric, harmonic and generalized logarithmic means of a and b.

Lemma 2.3 ([6]) $L_r(a,b)$ is increasing function on $(a,b) \in R_+^2$.

In this paper, we will prove that the function $F(a,b)$ is Schur-geometrically convex and Schur-harmonically convex on R_+^2.

3. Main Results

Theorem 3.1. Let $f: I \subseteq R_+ \to R$ and F be defined in Theorem A.

(i). If f is convex and increasing on I, then F is Schur-geometrically convex on I^2.

(ii). If f is concave and decreasing on I, then F is Schur-geometrically concave on I^2.

Proof. If $a, b \in I^\circ$ and $a = b$, we have $F(a,a) = f(a)$. For all $a, b \in I^\circ, a \ne b$, a straightforward computation gives

$$\begin{aligned} \frac{\partial F}{\partial b} &= \frac{1}{b-a} f(b) - \frac{1}{b-a} F(a,b), \\ \frac{\partial F}{\partial a} &= -\frac{1}{b-a} f(a) + \frac{1}{b-a} F(a,b). \end{aligned} \tag{3}$$

If f is convex and increasing on I, by the inequality (2), we obtain

$$\begin{aligned} &(b-a)\left(b\frac{\partial F}{\partial b} - a\frac{\partial F}{\partial a}\right) \\ &= af(a) + bf(b) - (a+b)F(a,b) \\ &= af(a) + bf(b) - \frac{a+b}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\ge \frac{1}{2}\left[2af(a) + 2bf(b) - (a+b)(f(a) + f(b))\right] \\ &= \frac{1}{2}(b-a)(f(b) - f(a)) \ge 0. \end{aligned} \tag{4}$$

Hence, $F(a,b)$ is Schur-geometrically convex on I^2. If f is concave and decreasing on I, then the inequality (4) is reversed. According to Lemma 2.1, it follows that $F(a,b)$ is Schur-geometrically concave I^2. This completes the proof of Theorem 3.1.

Theorem 3.2. Let $f: I \subseteq R_+ \to R$ and F be defined in Theorem A.

(i). If f is convex and increasing on I, then F is Schur-harmonically convex on I^2.

(ii). If f is concave and decreasing on I, then F is Schur-harmonically concave on I^2.

Proof. If $a, b \in I^\circ$ and $a = b$, we have $F(a,a) = f(a)$. For all $a, b \in I^\circ, a \ne b$, if f is convex and increasing, using inequality (3) and (2), we get

$$\begin{aligned} &(b-a)\left(b^2\frac{\partial F}{\partial b} - a^2\frac{\partial F}{\partial a}\right) \\ &= a^2 f(a) + b^2 f(b) - \frac{a^2+b^2}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\ge a^2 f(a) + b^2 f(b) - \frac{a^2+b^2}{2}\left(f(a) + f(b)\right) \\ &= \frac{1}{2}(b^2 - a^2)\left(f(b) - f(a)\right) \ge 0. \end{aligned} \tag{5}$$

Therefore, $F(a,b)$ is Schur-harmonically convex function on I^2. If f is concave and decreasing function on I, then the inequality (5) is reversed. According to Lemma 2.2, it follows that $F(a,b)$ is Schur-harmonically concave function on I^2. The proof of Theorem 3.2 is complete.

4. Applications

Theorem 4.1. For $a>0$ and $b>0$, if $r\ge 1$, then $L_r(a,b)$ is Schur-geometrically convex and Schur-harmonically convex.

Proof. Taking $f(x)=x^r$ for all $x\in R_+$, if $a\ne b$, it follows that

$$F(a,b)=\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x=\frac{1}{b-a}\int_a^b x^r\mathrm{d}x=\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}$$

and $f(x)=x^r$ is convex increasing on R_+ for $r\ge 1$. Therefore, by Theorem 3.1 and 3.2, we have

$$F(a,b)=\begin{cases}\dfrac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}, & a\ne b,\\ a^r, & a=b\end{cases}$$

is Schur-geometrically convex and Schur-harmonically convex on R_+^2 for $r\ge 1$, then $L_r(a,b)$ is Schur-geometrically convex and Schur-harmonically convex on R_+^2 for $r\ge 1$. Thus, Theorem 4.1 is proved.

Corollary 4.1.1. For $b>a>0$ and $r\ge 1$, define $u_a=ta+(1-t)b$, $v_a=(1-t)a+tb$, $u_g=a^tb^{1-t}$, $v_g=a^{1-t}b^t$, $u_h=ta^{-1}+(1-t)b^{-1}$, and $v_h=(1-t)a^{-1}+tb^{-1}$ for $t\in(0,1)$. Then

(1) when $t\in(0,1)$ and $t\ne 1/2$, we have

$$\left[\frac{u_h^{-(r+1)}-v_h^{-(r+1)}}{(r+1)(u_h^{-1}-v_h^{-1})}\right]^{1/r}\le\left[\frac{u_g^{r+1}-v_g^{r+1}}{(r+1)(u_g-v_g)}\right]^{1/r}$$
$$\le\left[\frac{u_a^{r+1}-v_a^{r+1}}{(r+1)(u_a-v_a)}\right]^{1/r}\le L_r(a,b);$$

(2) when $t=1/2$, we have

$$H(a,b)\le G(a,b)\le A(a,b)=L_r(a,b).$$

Proof. When $t=1/2$, it is easy to obtain that $H(a,b)\le G(a,b)\le L_r(a,b)$. When $t\in(0,1)$ and $t\ne 1/2$, by Corollary 2 in [6] and Lemma 2.3, Corollary 4.1.1 is thus proved.

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