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Backward Orbit Conjecture for Lattès Maps

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Abstract For a Lattès map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ defined over a number field K, we prove a conjecture on the integrality of points in the backward orbit of $P \in \mathbb{P}(\bar{K})$ under ϕ.

Keywords: *backward orbit conjecture, Lattès maps*

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1. Introduction

Let $\phi : \mathbb{P}^1 \to P^1$ be a rational map of degree ≥ 2 defined over a number field K, and write ϕ^n for the nth iterate of ϕ. For a point $P \in \mathbb{P}^1$, let $\phi^+(P) = \{P, \phi(P), \phi^2(P), ...\}$ be the forward orbit of P under ϕ, and let

$$\phi^-(P) = \bigcup_{n \geq 0} \phi^{-n}(P)$$

be the backward orbit of P under ϕ. We say P is ϕ−preperiodic if and only if $\phi^+(P)$ is finite.

Viewing the projective line $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$ and taking $P \in \mathbb{A}^1(K)$, a theorem of Silverman [4] states that if ∞ is not a fixed point for ϕ^2, then $\phi^+(P)$ contains at most finitely many points in $\mathcal{O}_K$, the ring of algebraic integers in K. If S is the set of all archimedean places for K, then $\mathcal{O}_K$ is the set of points in $\mathbb{P}^1(K)$ which are S-integral relative to ∞ (see section 2). Replacing ∞ with any point $Q \in \mathbb{P}^1(K)$ and S with any finite set of places containing all the archimedean places, Silverman's Theorem can be stated as: If Q is not a fixed point for ϕ^2, then $\phi^+(P)$ contains at most finitely many points which are S-integral relative to Q.

A conjecture for finiteness of integral points in backward orbits was stated in [[6], Conj. 1.2].

Conjecture 1.1. *If $Q \in \mathbb{P}^1(\bar{K})$ is not S-preperiodic, then $\phi^-(P)$ contains at most finitely many points in $\mathbb{P}^1(\bar{K})$ which are S-integral relative to Q.*

In [6], Conjecture 1.1 was shown true for the powering map $\phi(z) = z^d$ with degree $d \geq 2$, and consequently for Chebyschev polynomials. A gener-alized version of this conjecture, which is stated over a dynamical family of maps $|\varphi|$, is given in [[1], Sec. 4]. Along those lines, our goal is to prove a general form of Conjecture 1.1 where $|\varphi|$ is the family of Lattès maps associate to a fixed elliptic curve E defined over K (see Section 3).

2. The Chordal Metric and Integrality

2.1. The Chordal Metric on $\mathbb{P}^N$. Let M_K be the set of places on K normalized so that the product formula holds: for all $\alpha \in K^*$,

$$\prod_{v \in M_K} |\alpha|_v = 1.$$

For points $P = [x_0 : x_1 : \cdots : x_N]$ and $Q = [y_0 : y_1 : \cdots : y_N]$ in $\mathbb{P}^N(\bar{K}_v)$, define *the v-adic chordal metric as*

$$\Delta_v(P,Q) = \frac{\max_{i,j}\left(\left|x_i y_j - x_j y_i\right|_v\right)}{\max_i\left(\left|x_i\right|_v\right).\max_i\left(\left|y_i\right|_v\right)}.$$

Note that Δ_v is independent of choice of projective coordinates for P and Q, and $0 \leq \Delta_v(\cdot,\cdot) \leq 1$ (see [2]).

2.2. Integrality on Projective Curves. Let C be an irreducible curve in $\mathbb{P}^N$ defined over K and S a finite subset of M_K which includes all the archimedean places. A divisor on C defined over $\bar{K}$ is a finite formal sum $\sum n_i Q_i$ with $n_i \in \mathbb{Z}$ and $Q_i \in C(\bar{K})$. The divisor is effective if $n_i > 0$ for each i, and its support is the set Supp(D) = $\{Q_1, \cdots, Q_\ell\}$.

Let $\lambda_{Q,v}(P) = -\log \Delta_v(P,Q)$ and $\lambda_{D,v}(P) = \sum n_i \lambda_{Q_i,v}(P)$ when $D = \sum n_i Q_i$. This makes $\lambda_{D,v}$ an arithmetic distance function on C (see [3]) and as with any arithmetic distance function, we may use it to classify the integral points on C.

For an effective divisor $D = \sum n_i Q_i$ on C defined over $\bar{K}$, we say $P \in C(\bar{K})$ is S-integral relative to D, or P is a (D, S)-integral point, if and only if $\lambda_{Q_i^\sigma,v}(P^\tau) = 0$ for all embeddings $\sigma, \tau : K \to \bar{K}$ and for all places $v \notin S$. Furthermore, we say the set $\mathcal{R} \subset C(\bar{K})$ is S-integral relative to D if and only if each point in $\mathcal{R}$ is S-integral relative to D.

As an example, let C be the projective line $\mathbb{A}^1 \cup \{\infty\}$, S be the Archimedean place of $K = \mathbb{Q}$, and $D = \infty$. For $P = x/y$, with x and y are relatively prime in $\mathbb{Z}$, we have $\lambda_{D,v}(P) = -\log|y|_v$ for each prime v. Therefore, P is S-integral relative to D if and only if $y = \pm 1$; that is, P is S-integral relative to D is and only if $P \in \mathbb{Z}$.

From the definition we find that if $S_1 \subset S_2$ are finite subsets of M_K which contains all the archimedean places, then P is a (D, S_2)-integral point implies that P is a (D, S_1)-integral point. Similarly, if Supp $(D_1) \subset$ Supp(D_2), then P is a (D_2, S)-integral point implies that P is also a (D_2, S)-integral point. Therefore enlarging S or Supp(D) only enlarges the set of (D, S)-integrals points on $C(\bar{K})$.

For $\phi : C_1 \to C_2$ a finite morphism between projective curves and $P \in C_2$, write

$$\phi^* P = \sum_{Q \in \phi^{-1}(P)} e_\phi(Q) \cdot Q$$

where $e_\phi(Q) \geq 1$ is the ramification index of ϕ at Q. Furthermore, if $D = \sum n_i Q_i$ is a divisor on C, then we define $\phi^* D = \sum n_i \phi^* Q_i$.

Theorem 2.1 *(Distribution Relation). Let $\varphi : C_1 \to C_2$ be a finite mor-phism between irreducibly smooth curves in $\mathbb{P}^N(\bar{K})$. Then for $Q \in C_1$, there is a finite set of places S, depending only on φ and containing all the archimedean places, such that $\lambda_{P,v} \circ \varphi = \lambda_{\varphi^* P,v}$ for all $v \notin S$.*

Proof. See [[3], Prop. 6.2b] and note that for projective varieties the $\lambda_{\delta W \times V}$ term is not required, and that the big-O constant is an M_K-bounded constant not depending on P and Q.

Corollary 2.2. *Let $\varphi : C_1 \to C_2$ be a finite morphism between irreducibly smooth curves in $\mathbb{P}^N(\bar{K})$, let $P \in C_1(\bar{K})$, and let D be an effective divisor on C_2 defined over K. Then there is a finite set of places S, depending only on φ and containing all the archimedean places, such that $\phi(P)$ is S-integral relative to D if and only P is S-integral relative to $\phi^* D$.*

Proof. Extend S so that the conclusion of Theorem 2.1 holds. Then for $D = \sum n_i Q_i$ with each $n_i > 0$ and $Q_i \in C_2(\bar{K})$, we have that.

$$\lambda_{\phi^* D,v}(P) = \lambda_{D,v}(\phi(P)) = \sum n_i \lambda_{Q_i,v}(\phi(P)).$$

So $\lambda_{\phi^* D,v}(P) = 0$ if and only if $\lambda_{Q_i,v}(\phi(P)) = 0$.

3. Main Result

Let E be an elliptic curve, $\psi : E \to E$ a morphism, and $\pi : E \to \mathbb{P}^1$ be a finite covering. A *Lattès map* is a rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ making the following diagram commute:

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^1 & \xrightarrow{\phi} & \mathbb{P}^1 \end{array}$$

For instance, if E is defined by the Weierstrass equation $y^2 = x^3 + ax^2 + bx + c$, $\psi = [2]$ is the multiplication-by-2 endomorphism on E, and $\pi(x,y) = x$, then

$$\phi(x) = \frac{x^4 - 2bx^2 + 8cx + b^2 - 4ac}{4x^3 + 4ax^2 + 4bx + 4c}.$$

Fix an elliptic curve E defined over a number field K, and for $P \in \mathbb{P}^1(\bar{K})$ define:

$$[\varphi] = \left\{ \phi : \mathbb{P}^1 \to \mathbb{P}^1 \,\middle|\, \begin{array}{l} \text{there exist } K-\text{morphosm } \psi : E \to E \\ \text{and finite covering } \pi : E \to \mathbb{P}^1 \text{ such} \\ \text{that } \pi \circ \psi = \phi \circ \pi \end{array} \right\}$$

$$\Gamma_0 = \bigcup_{\phi \in [\varphi]} \phi^+(P)$$

$$\Gamma = \left(\bigcup_{\phi \in [\varphi]} \phi^+(\Gamma_0) \right) \cup \mathbb{P}^1(\bar{K})_{[\varphi]-preper}$$

A point Q is $[\varphi]$-preperiodic if and only if Q is ϕ-preperiodic for some $\phi \in [\varphi]$. We write $\mathbb{P}^1(\bar{K})_{[\varphi]-preper}$ for the set of $[\varphi]$-preperiodic points in $\mathbb{P}^1(\bar{K})$.

Theorem 3.1. *If $Q \in \mathbb{P}^1(\bar{K})$ is not $[\varphi]$-periodic, then Γ contains at most finitely many points in $\mathbb{P}^1(\bar{K})$ which are S-integral relative to Q.*

Proof. Let Γ_0' *be the End(E)-submodule of* $E(\bar{K})$ *that is finitely generated by the points in* $\pi^{-1}(P)$, *and let*

$\Gamma' = \{\xi \in E(\bar{K}) | \lambda(\xi) \in \Gamma_0' \text{ for some non-zero } \lambda \in \text{End}(E)\}.$

Then $\pi^{-1}(\Gamma) \subset \Gamma'$. Indeed, if $\pi(\xi) \in \Gamma$ is not $[\varphi]$-preperiodic, then ξ is non torsion and $(\phi_1 \circ \pi)(\xi) \in \Gamma_0$ for some Lattès map ϕ_1. So $(\phi_1 \circ \pi)(\xi) \in \Gamma_0$ for some morphism $\psi_1 : E \to E$, and this gives $(\pi \circ \psi_1)(\xi) \in \phi_2(P)$ for some Lattès map ϕ_2. Therefore $\psi_1(\xi) \in (\pi^{-1} \circ \phi_2)(P)$ $= (\psi_2 \circ \pi^{-1})(P)$ for some morphism $\psi_2 : E \to E$. Since any morphism $\psi : E \to E$ is of the form $\psi(X) = \alpha(X) + T$ where $\alpha \in End(E)$ and $T \in E_{tors}$ (see [[5], 6.19]), we find that there is a $\lambda \in End(E)$ such that $\lambda(\xi)$ is in Γ_0', the End(E)-submodule generated by $\pi^{-1}(P)$. Otherwise, if $\pi(\xi) \in \Gamma$ is $[\varphi]$-preperiodic, then $\pi(E(\bar{K})_{tors}) = \mathbb{P}^1(\bar{K})_{[\varphi]-preper}$ ([[5], Prop. 6.44]) gives that ξ may be a torsion point; again $\xi \in \Gamma'$ since $E(\bar{K})_{tors} \subset \Gamma'$. Hence $\pi^{-1}(\Gamma) \subset \Gamma'$.

Let D be an effective divisor whose support lies entirely in $\pi^{-1}(Q)$, let $\mathcal{R}_Q$ be the set of points in Γ which are S-integral relative to Q, and let $\mathcal{R}_D'$ be the set of points in Γ' which are S-integral relative to D. Extending S so that Theorem 2.1 holds for the map $\pi : E \to \mathbb{P}^1$, and since Supp(D) $\subset$ Supp$(\pi^* D)$, we have: if $\gamma \in \Gamma$ is S-integral relative to Q, then $\pi^{-1}(\gamma)$ is S-integral relative to D. Therefore $\pi^{-1}(\mathcal{R}_Q) \subset \mathcal{R}_D'$. Now π is a finite map and $\pi(E(\bar{K})) = \mathbb{P}^1(\bar{K})$; so to complete the proof, it suffices to show that D can be chosen so that $\mathcal{R}_D'$ is finite.

From [[5], Prop. 6.37], we find that if Λ is a nontrivial subgroup of Aut(E), then $E/\Lambda \cong \mathbb{P}^1$ and the map $\pi : E \to \mathbb{P}$ can be determine explicitly. The four possibilities for π, which are $\pi(x, y) = x, x^2, x^3$, or y correspond respectively to the four possibilities for Λ, which are $\Lambda = \mu_2, \mu_4, \mu_6$, or μ_3, which in turn depends only on the j-invariant of E. (Here, μ_N denotes the Nth roots of unity in $\mathbb{C}$.)

First assume that $\pi(x, y) \neq y$. Since Q is not [']-preperiodic, take $\xi \in \pi^{-1}(Q)$ to be non torsion. Then $-\xi \in \pi^{-1}(Q)$ since $\Lambda = \mu_2, \mu_4, or\ \mu_6$, and $\xi - (-\xi) = 2\xi$ is non-torsion. Taking $D = (\xi) + (-\xi)$, [[1], Thm. 3.9(i)] gives that $\mathcal{R}_D'$ is finite.

Suppose that $\pi(x, y) = y$. Then $\pi(x, y) = \{\xi, \xi', \xi''\}$ where $\xi + \xi' + \xi'' = 0$ and ξ is non-torsion since Q is not $[\varphi]$-preperiodic. Assuming that both $\xi - \xi'$ and $\xi - \xi''$ are torsion give that 3ξ is torsion, and this contradicts the fact that ξ is torsion. Therefore, we may assume that $\xi - \xi'$ is non-torsion. Now taking $D = (\xi) + (\xi')$, [[1], Thm. 3.9(i)] again gives that $\mathcal{R}_D'$ is finite. Hence RQ, the set of points in Γ which are S-integral relative to Q, is finite.

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