Turkish Journal of Analysis and Number Theory, 2015, Vol. 3, No. 2, 37-42
Available online at http://pubs.sciepub.com/tjant/3/2/1

DOI:10.12691/tjant-3-2-1

The Rogers-Ramanujan Identities

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Abstract In 1894, Rogers found the two identities for the first time. In 1913, Ramanujan found the two identities later and then the two identities are known as The Rogers-Ramanujan Identities. In 1982, Baxter used the two identities in solving the Hard Hexagon Model in Statistical Mechanics. In 1829 Jacobi proved his triple product identity; it is used in proving The Rogers-Ramanujan Identities. In 1921, Ramanujan used Jacobi's triple product identity in proving his famous partition congruences. This paper shows how to generate the generating function for $C'(n)$, $C_1'(n)$, $C''(n)$ and $C_1''(n)$, and shows how to prove the Corollaries 1 and 2 with the help of Jacobi's triple product identity. This paper shows how to prove the Remark 3 with the help of various auxiliary functions and shows how to prove The Rogers-Ramanujan Identities with help of Ramanujan's device of the introduction of a second parameter a.

Keywords: *at most, auxiliary function, convenient, expansion, minimal difference, operator, Ramanujan's device*

Cite This Article: Fazlee Hossain, Sabuj Das, and Haradhan Kumar Mohajan, "The Rogers-Ramanujan Identities." *Turkish Journal of Analysis and Number Theory*, vol. 3, no. 2 (2015): 37-42. doi: 10.12691/tjant-3-2-1.

1. Introduction

In this article, we give some related definitions of $P(n)$, $C'(n)$, $P_m(n-m^2)$, $C_1'(n)$, $C''(n)$, $P_m(n-m(m+1))$ and $C_1''(n)$. We describe the generating functions for $C'(n)$, $P_m(n-m^2)$, $C_1'(n)$, $C''(n)$, $P_m(n-m(m+1))$ and $C_1''(n)$, and establish the Remarks 1 and 2 with numerical examples and also prove the Corollaries 1 and 2 with the help of Jacobi's triple product identity [3]. We transfer the auxiliary function into another auxiliary function with the help of Ramanujan's device of the introduction of a second parameter *a* *[5]*,

i.e.,

$$G_k(a,x)=\prod_{n=0}^{\infty}(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right) C_n$$

to

$$G_1(x,x)=\sum_{m=0}^{\infty}\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)\left(1-x^{5m+5}\right),$$

where $k=1$, and $a=x$, it is used in proving The Rogers-Ramanujan Identity 1. We prove The Rogers-Ramanujan Identities with the help of auxiliary functions.

2. Some Related Definitions

$P(n)$ [7]: The number of partitions of *n* like: 4, 3+1, 2+2, 2+1+1, 1+1+1+1 $\therefore P(4)=5$.

$C'(n)$ [6]: The number of partitions of *n* into parts each of which is of one of the forms $5m+1$ and $5m+4$.

$P_m(n-m^2)$: The number of partitions of $n-m^2$ into *m* parts at most.

$C''(n)$: The number of partitions of *n* into parts of the forms $5m+2$ and $5m+3$.

$C_1'(n)$: The number of partitions of *n* into parts without repetitions or parts whose minimal difference is 2.

$P_m(n-m(m+1))$: The number of partitions of $n-m(m+1)$ into *m* parts at most.

$C_1''(n)$: The number of partitions of *n* into parts not less than 2 and with minimal difference 2.

3. Generating Functions for $C'(n)$ and $C''(n)$

In this section we describe the generating functions for $C'(n)$ and $C''(n)$ respectively. The generating function for $C'(n)$ is of the form [5];

$$\sum_{m=0}^{\infty} \frac{1}{\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)}$$
$$= \frac{1}{(1-x)\left(1-x^4\right)\left(1-x^6\right)\left(1-x^9\right)\ldots\infty} \quad (1)$$
$$= 1 + x + x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + \ldots\infty$$
$$= 1 + \sum_{n=1}^{\infty} C'(n)\, x^n$$

where the coefficient $C'(n)$ of x^n is the number of partitions of n into parts each of which is of one of these forms $5m + 1$ and $5m + 4$.

Now we consider a special function, which is given below:

$$\frac{x^{m^2}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)}$$
$$= x^{m^2} \sum_{n=m^2}^{\infty} P_m\left(n-m^2\right) x^{n-m^2}$$
$$= \sum_{n=m^2}^{\infty} P_m\left(n-m^2\right) x^n$$

It is convenient to define $P_m(0) = 1$. The coefficient $P_m\left(n-m^2\right)$ of x^n in the above expansion is the number of partitions of $n-m^2$ into m parts at most. Another special function, which is defined as;

$$1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)}$$
$$= 1 + \frac{x}{1-x} + \frac{x^4}{(1-x)\left(1-x^2\right)}$$
$$+ \frac{x^9}{(1-x)\left(1-x^2\right)\left(1-x^3\right)} + \ldots\infty \quad (2)$$
$$= 1 + x + x^2 + x^3 + 2x^4 + 2x^5$$
$$+ 3x^6 + 3x^7 + \ldots\infty$$
$$= 1 + \sum_{n=1}^{\infty} C_1'(n)\, x^n$$

where the coefficient $C_1'(n)$ is the number of partitions of n into parts without repetitions or parts, whose minimal difference is 2.

From (1) and (2) we can establish the following Remark:

Remark 1:

$$C_1'(11) = C'(11) \quad (3)$$

i.e., the number of partitions of n with minimal difference 2 is equal to the number of partitions of n into parts of the forms $5m + 1$ and $5m + 4$.

Example 1: For $n = 11$, there are 7 partitions of 11 that are enumerated by $C_1'(n)$ of above statement, which are given bellow [6]:

$11, 10+1, 9+2, 8+3, 7+4, 7+3+1, 6+4+1,$

$\therefore\ C_1'(11) = 7.$

There are 7 partitions of 11 are enumerated by $C_1'(n)$ of above statement, which are given bellow:

$11, 9+1+1, 6+4+1, 6+1+1+1+1+1,$

$4+4+1+1+1, 4+1+1+1+1+1+1+1,$

$1+1+1+1+1+1+1+1+1+1+1,$

$\therefore\ C'(11) = 7.$

Hence, $C_1'(11) = C'(11)$.

We can conclude that, $C_1'(11) = C'(11)$.

$$1 + \sum_{n=1}^{\infty} C'(n)\, x^n = 1 + \sum_{n=1}^{\infty} C_1'(n)\, x^n.$$
$$1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)}$$
$$= \sum_{m=0}^{\infty} \frac{1}{\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)},$$

which will be proved later as identity 1, it is known as The Rogers-Ramanujan identity 1.

The generating function for $C''(n)$ is of the form [1];

$$\sum_{m=0}^{\infty} \frac{1}{\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)}$$
$$= \frac{1}{\left(1-x^2\right)\left(1-x^3\right)\left(1-x^7\right)\left(1-x^8\right)\ldots\infty} \quad (4)$$
$$= 1 + 0.x + x^2 + x^3 + x^4 + x^5 + 2x^6 + 2x^7 + \ldots\infty$$
$$= 1 + \sum_{n=1}^{\infty} C''(n)\, x^n$$

where the coefficient $C''(n)$ is the number of partitions of n into parts of the forms $5m + 2$ and $5m+3$.

Now we consider a special function, which is of the form [1];

$$\frac{x^{m(m+1)}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)}$$
$$= x^{m(m+1)} \sum_{n=m(m+1)}^{\infty} P_m\left(n-m(m+1)\right) x^{n-m(m+1)}$$
$$= \sum_{n=m(m+1)}^{\infty} P_m\left(n-m(m+1)\right) x^n,$$

where the coefficient $P_m\left(n-m(m+1)\right)$ of x^n in the above expansion is the number of partitions of $n-m(m+1)$ into m parts at most.

Another special function, which is defined as;

$$1+\sum_{m=1}^{\infty}\frac{x^{m(m+1)}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)}$$
$$=1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)\left(1-x^2\right)}$$
$$+\frac{x^{12}}{(1-x)\left(1-x^2\right)\left(1-x^3\right)}+\ldots\infty \quad (5)$$
$$=1+x^2+x^3+x^4+x^5+2x^6$$
$$+2x^7+3x^8+\ldots\infty$$
$$=1+\sum_{n=1}^{\infty}C_1''(n)\,x^n,$$

where the coefficient $C_1''(n)$ is the number of partitions of n into parts not less than 2 and with minimal difference 2.

From (4) and (5) we can establish the following Remark:

Remarks 2:

$$C_1''(n)=C''(n), \quad (6)$$

i.e., the number of partitions of n into parts not less than 2 and with minimal difference 2 is equal to the number of partitions of n into parts of the forms $5m+2$ and $5m+3$.

Example 2: If $n = 11$, the four partitions of 11 into parts not less than 2 and with minimal difference 2 are given below:

$$11,\ 9+2,\ 8+3,\ 7+4.$$

Hence, $C_1''(11)=4$.

Again the four partitions of 11 into parts of the form $5m+2$ and $5m+3$ are given as;

$$8+3,\ 7+2+2,\ 3+3+3+2,\ 3+2+2+2+2.$$

Hence, $C''(11)=4$.

$\therefore\ C_1''(11)=C''(11)$.

We can conclude that, $C_1''(n)=C''(n)$.

i.e., $1+\sum_{m=1}^{\infty}C_1''(n)\,x^n=1+\sum_{m=1}^{\infty}C''(n)\,x^n$

$$1+\sum_{m=1}^{\infty}\frac{x^{m(m+1)}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)}$$
$$=\prod_{m=0}^{\infty}\frac{1}{\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)},$$

which will be proved later as identity 2, it is known as The Rogers-Ramanujan identity 2.

Now we give two Corollaries, which are related to the Jacobi's triple product identity [3].

Corollary 1:

$$\prod_{n=0}^{\infty}\left(1-x^{5n+1}\right)\left(1-x^{5n+4}\right)\left(1-x^{5n+5}\right)$$
$$=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+3)}{2}}.$$

Proof: From Jacobi's Theorem [2] we have;

$$\prod_{n=0}^{\infty}\left\{\left(1-x^{2n}\right)\left(1+x^{2n+1}z\right)\left(1+x^{2n-1}z^{-1}\right)\right\}$$
$$=\sum_{n=-\infty}^{\infty}x^{n^2}z^n,$$

for all z except $z = 0$, if $|x|<1$.

If we write $x^{5/2}$ for x, $-x^{3/2}$ for z and replace n by $n + 1$ on the left hand side we obtain;

$$\prod_{n=0}^{\infty}\left(1-x^{5n+1}\right)\left(1-x^{5n+4}\right)\left(1-x^{5n+5}\right)$$
$$=1-x-x^4+x^7+x^{13}-\ldots\infty$$
$$=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+3)}{2}}.$$

Hence, the Corollary.

Corollary 2:

$$\prod_{n=0}^{\infty}\left(1-x^{5n+2}\right)\left(1-x^{5n+3}\right)\left(1-x^{5n+5}\right)$$
$$=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+1)}{2}}.$$

Proof: From Jacobi's Theorem we have;

$$\prod_{n=0}^{\infty}\left(1-x^{2n}\right)\left(1+x^{2n+1}z\right)\left(1+x^{2n-1}z^{-1}\right)$$
$$=\sum_{n=-\infty}^{\infty}x^{n^2}z^n,$$

for all z except $z = 0$, when $|x|<1$.

If we write $x^{5/2}$ for x, $-x^{1/2}$ for z and replace n by $n + 1$ on the left hand side we obtain;

$$\prod_{n=0}^{\infty}\left(1-x^{5n+2}\right)\left(1-x^{5n+3}\right)\left(1-x^{5n+5}\right)$$
$$=1-x^2-x^3+x^9+x^{11}-\ldots\infty$$
$$=\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(5n+1)}{2}}.$$

Hence the Corollary.

4. The Rogers-Ramanujan Identities

First we transfer the following auxiliary function into another auxiliary function. Let us consider the auxiliary function [1, 2] with $|x|<1$ and $|a|<1$.

$$G_k(a,x)=\prod_{n=0}^{\infty}(-1)^n a^{2n}x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right)C_n \quad (7)$$

it is known as Ramanujan's device of the introduction of a second parameter a, where k is 0, 1 or 2 and $C_0=1$,

$$C_n = \frac{(1-a)(1-ax)\dots(1-ax^{n-1})}{(1-x)(1-x^2)\dots(1-x^n)}.$$

Hence,

$$G_k(a,x) = \sum_{n=0}^{\infty}\left[\begin{array}{l}(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right)\\ \times\dfrac{(1-a)(1-ax)\dots(1-ax^{n-1})}{(1-x)(1-x^2)\dots(1-x^n)}\end{array}\right].$$

$$\frac{G_k(a,x)}{(1-a)(1-ax)\dots\infty}$$

$$= \sum_{n=0}^{\infty}\left[\begin{array}{l}(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}}\\ \times\dfrac{1-a^k x^{2kn}}{\left[\begin{array}{l}(1-x)(1-x^2)\dots(1-x^n)\\ \times(1-ax^n)(1-ax^{n+1})\dots\infty\end{array}\right]}\end{array}\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right)P_n Q_n(a)$$

where $P_n = \prod_{r=1}^{n}\frac{1}{1-x^r}$,

$$Q_n(a) = \prod_{r=n}^{\infty}\frac{1}{1-ax^r} = H_k(a,x) \qquad (8)$$

which is another auxiliary function, and it is used in proving The Rogers-Ramanujan Identities [1].

But from (7) we can easily verify that with k = 1, 2 and $a = x$.

$$G_1(x,x) = 1 - x - x^4 + x^7 + x^{13} - \dots\infty$$

$$G_1(x,x) = \prod_{n=0}^{\infty}\left(1-x^{5n+1}\right)\left(1-x^{5n+4}\right)\left(1-x^{5n+5}\right) \qquad (9)$$

(by Corollary 1).

$$G_2(x,x) = 1 - x^2 - x^3 + x^9 + x^{11} - \dots\infty$$

$$G_2(x,x) = \prod_{m=0}^{\infty}\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)\left(1-x^{5m+5}\right) \qquad (10)$$

(by Corollary 2).

From (8) we can also find that, if k =1 and $a = x$, then;

$$H_1(x,x) = \frac{G_1(x,x)}{(1-x)(1-x^2)(1-x^3)\dots\infty}$$

$$= \frac{\prod_{m=0}^{\infty}\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)\left(1-x^{5m+5}\right)}{(1-x)(1-x^2)(1-x^3)\dots\infty} \qquad (11)$$

$$= \prod_{m=0}^{\infty}\frac{1}{\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)}.$$

Again for k = 2 and $a = x$, we get;

$$H_2(x,x) = \frac{G_2(x,x)}{(1-x)(1-x^2)(1-x^3)\dots\infty}$$

$$= \frac{\prod_{m=0}^{\infty}\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)\left(1-x^{5m+5}\right)}{(1-x)(1-x^2)(1-x^3)\dots\infty} \qquad (12)$$

$$= \prod_{m=0}^{\infty}\frac{1}{\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)}.$$

Now we can consider the following Remark [2].

Remark 3: $H_k - H_{k-1} = a^{k-1}\eta H_{3-k}$, where the operator η is defined by $\eta f(a) = f(ax)$, and k = 1 or 2.

Proof: From (8) we have;

$$H_k = H_k(a,x)$$

$$= \sum_{n=0}^{\infty}(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right)P_n Q_n(a),$$

where $P_n = \prod_{r=1}^{n}\frac{1}{1-x^r}$, and $Q_n(a) = \prod_{r=n}^{\infty}\frac{1}{1-ax^r}$,

It is convenient to define $P_0 = 1$, $H_0 = 1$. We have;

$$H_k - H_{k-1}$$

$$= \sum_{n=0}^{\infty}\left\{\begin{array}{l}(-1)^n a^{2n} x^{\frac{n(5n+1)}{2}}\\ \times\left[\begin{array}{l}x^{-kn} - a^k x^{kn} - x^{(1-k)n}\\ +a^{k-1}x^{n(k-1)}\end{array}\right]P_n Q_n\end{array}\right\}$$

$$= \sum_{n=0}^{\infty}(-1)^n a^{2n} x^{\frac{n(5n+1)}{2}} \times\left[\begin{array}{l}a^{k-1}x^{n(k-1)}\left(1-ax^n\right)\\ +x^{-kn}\left(1-x^n\right)\end{array}\right]P_n Q_n.$$

Now we have, $\left(1-ax^n\right)Q_n = Q_{n+1}$ and $\left(1-x^n\right)P_n = P_{n-1}$, hence,

$$H_k - H_{k-1}$$

$$= \sum_{n=0}^{\infty}(-1)^n a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} P_n Q_{n+1}$$

$$+ \sum_{n=0}^{\infty}(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}} P_{n-1} Q_n.$$

In the second sum on the right hand side of the Identity we change n into n + 1. Thus,

$$H_k - H_{k-1}$$

$$= \sum_{n=0}^{\infty}(-1)^n a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} P_n Q_{n+1}$$

$$- \sum_{n=0}^{\infty}(-1)^n a^{2(n+1)} x^{\frac{(n+1)(5n+6)-2k(n+1)}{2}} P_n Q_{n+1}.$$

$$= \sum_{n=0}^{\infty} (-1)^n \begin{Bmatrix} a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} \\ -a^{2(n+1)} x^{\frac{(n+1)(5n+6)-2k(n+1)}{2}} \end{Bmatrix} P_n Q_{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \begin{Bmatrix} a^{2n+k-1} x^{\frac{n(5n+1)+2n(k-1)}{2}} \\ \otimes \left(1 - a^{3-k} x^{(2n+1)(3-k)}\right) \end{Bmatrix} P_n Q_{n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[a^{k-1} \eta \begin{Bmatrix} a^{2n} x^{\frac{n(5n+1)-2n(3-k)}{2}} \\ \times \left(1 - a^{3-k} x^{2n(3-k)}\right) \end{Bmatrix} \right] P_n Q_{n+1}.$$

We have $Q_{n+1} = \eta Q_n$ and so,

$$H_k - H_{k-1}$$

$$= a^{k-1} \eta \sum_{n=0}^{\infty} (-1)^n a^{2n} x^{\frac{n(5n+1)-2n(3-k)}{2}} \left(1 - a^{3-k} x^{2n(3-k)}\right) P_n Q_n$$

$$= a^{k-1} \eta H_{3-k}.$$

Hence, the Remark.

The Rogers-Ramanujan Identities

Identity 1 [4]:

$$1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)} = \prod_{m=0}^{\infty} \frac{1}{\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)}.$$

Identity 2 [4]:

$$1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)} = \prod_{m=0}^{\infty} \frac{1}{\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)}.$$

Proof: From (8) we have;

$$H_k(a,x) = \frac{G_k(a,x)}{(1-a)(1-ax)\ldots\infty} \qquad (13)$$

where $H_0 = 0$.

From above Remark we have;

$$H_k - H_{k-1} = a^{k-1} \eta H_{3-k}$$

where the operator η is defined by $\eta f(a) = f(ax)$, and $k = 1$ or 2. In particular

$$H_1 = \eta H_2,$$

$$H_2 - H_1 = a \eta H_1. \qquad (14)$$

So we have,

$$H_2 = \eta H_2 + a \eta^2 H_2. \qquad (15)$$

Suppose now that;

$$H_2 = 1 + c_1 a + c_2 a^2 + \ldots\infty. \qquad (16)$$

where the coefficients depend on x only. Substituting this into (15), we obtain;

$$1 + c_1 a + c_2 a^2 + \ldots\infty$$
$$= 1 + c_1 ax + c_2 a^2 x^2 + \ldots\infty + a\left(1 + c_1 ax^2 + c_2 a^2 x^4 + \ldots\infty\right).$$

Hence, equating the coefficients of various powers of a from both sides we get;

$$c_1 = \frac{1}{1-x}, c_2 = \frac{x^2}{1-x^2} c_1, c_3 = \frac{x^4}{1-x^3} c_2, \cdots,$$
$$c_n = \frac{x^{n(n-1)}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^n\right)}.$$

From (13) and (16), we have for $k = 2$;

$$\frac{G_2(a,x)}{(1-a)(1-ax)\ldots\infty} = H_2(a,x)$$
$$= 1 + \frac{a}{1-x} + \frac{a^2 x^2}{(1-x)\left(1-x^2\right)} + \frac{a^3 x^6}{(1-x)\left(1-x^2\right)\left(1-x^3\right)} + \ldots\infty.$$

If $a = x$, then;

$$1 + \frac{x}{1-x} + \frac{x^4}{(1-x)\left(1-x^2\right)} + \frac{x^9}{(1-x)\left(1-x^2\right)\left(1-x^3\right)} + \ldots\infty = \frac{G_2(x,x)}{(1-x)\left(1-x^2\right)\ldots\infty}.$$

Therefore,

$$1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x)\left(1-x^2\right)\ldots\left(1-x^m\right)} = \prod_{m=0}^{\infty} \frac{1}{\left(1-x^{5m+1}\right)\left(1-x^{5m+4}\right)}.$$

Hence the Identity 1.

Again from (13), (14) and (16) we have with $k = 1$,

$$\frac{G_1(a,x)}{(1-a)(1-ax)\ldots\infty} = H_1(a,x) = \eta H_2(a,x)$$
$$= 1 + \frac{ax}{1-x} + \frac{a^2 x^4}{(1-x)\left(1-x^2\right)} + \frac{a^3 x^9}{(1-x)\left(1-x^2\right)\left(1-x^3\right)} + \ldots\infty.$$

If $a = x$, then we have;

$$1+\frac{x^2}{1-x}+\frac{x^6}{(1-x)\left(1-x^2\right)}+\frac{x^{12}}{(1-x)\left(1-x^2\right)\left(1-x^3\right)}+\ ...\ \infty=\frac{G_1(x,x)}{(1-x)\left(1-x^3\right)\ ...\ \infty}.$$

Therefore,

$$1+\sum_{m=1}^{\infty}\frac{x^{m(m+1)}}{(1-x)\left(1-x^2\right)...\left(1-x^m\right)}=\prod_{m=0}^{\infty}\frac{1}{\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)}.$$

Hence the Identity 2.

5. Conclusion

In this study, we have shown $C_1'(n)=C'(n)$ with the help of a numerical example when n=11, and also have shown $C_1''(n)=C''(n)$ with the help of a numerical example when n =11. We have transferred the auxiliary function into another auxiliary function with the help of Ramanujan's device of the introduction of a second parameter *a*,
i.e.,

$$G_k(a,x)=\prod_{n=0}^{\infty}(-1)^n a^{2n} x^{\frac{n(5n+1)-2kn}{2}}\left(1-a^k x^{2kn}\right) C_n$$

to

$$G_2(x,x)=\sum_{m=0}^{\infty}\left(1-x^{5m+2}\right)\left(1-x^{5m+3}\right)\left(1-x^{5m+5}\right),$$

where k =2, and a = x, it is used in proving The Rogers-Ramanujan Identity 2. Finally we have proved The Roger-Ramanujan Identities with the help of auxiliary function,

$$H_k(a,x)=\frac{G_k(a,x)}{(1-a)(1-ax)\ ...\ \infty},$$

where $H_0=0$.

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