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A Double Inequality for the Harmonic Number in Terms of the Hyperbolic Cosine

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Abstract In the paper, the author present an inequality for bounding the harmonic number in terms of the hyperbolic cosine.

Keywords: *Inequality, Euler-Mascheroni constant, Harmonic number, Hyperbolic cosine*

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1. Introduction

The harmonic number H_n is defined as

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

and it has the following close connections with the Euler-Mascheroni constant γ:

$$\gamma = \lim_{n\to\infty}(H_n - \ln n) = 0.57721...$$

and

$$H_n = \psi(n+1) + \gamma,$$

where $\psi(x)$ is the digamma function which is the logarithmic derivative of the classical Euler gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \ \Re(z) > 0.$$

The harmonic number H_n has interesting applications in many areas of mathematics, such as number theory, special functions, and combinatorics. For example, Lagarias proved that the Riemann hypothesis is equivalent to the statement that

$$\sigma(n) \le H_n + e^{H_n}\ln H_n$$

for $n \in \mathbb{N}$, where $\sigma(n)$ denotes the sum of the divisors of n.

In [20], Paule and Schneider obtained the identity

$$\sum_{k=0}^{n}\binom{k}{n}^2 H_n = \binom{n}{2n}(2H_n - H_{2n}).$$

In [2], Alzer presented the inequality

$$\alpha\frac{\ln(\ln n+\gamma)}{n^2} \le H_n^{1/n} - H_{n+1}^{1/(n+1)} < \beta\frac{\ln(\ln n+\gamma)}{n^2}$$

for $n \ge 2$, where the constants $\alpha =$ 0.0140… and $\beta =$ 1 are the best possible. In [5], Batir gave an inequality

$$\ln\frac{\pi^2}{6} - \ln\left(e^{1/(n+1)}-1\right) < H_n < \gamma - \ln\left(e^{1/(n+1)}-1\right).$$

This double inequality was refined in [4] by replacing $\ln\frac{\pi^2}{6}$ by 1. It also inspired Mortici to construct a sequence

$$\mu_n = \sum_{k=1}^{n}\frac{1}{k} + \ln\left(e^{a/(n+b)}-1\right) - \ln a$$

in [15], which converges to more quickly.

For more information on the harmonic number H_n, please refer to [2,6-19,21-26] and plenty of references therein.

In this paper, we will establish a new double inequality for bounding the harmonic number H_n in terms of the hyperbolic cosine.

Our main result may be stated as the following theorem.

Theorem 1.1. For all positive integers $n \in \mathbb{N}$, we have

$$\alpha \le H_n - \ln n - \ln\cosh\frac{1}{\sqrt{n}} < \beta, \qquad (1.1)$$

where the constants $\alpha = 1 - \ln(\cosh 1) =$ 0.5662… and $\beta = \gamma =$ 0.5772… are the best impossible.

2. Lemmas

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 ([[3], p. 384]). Let $n \geq 1$ and $k \geq 0$ be integers, for $x > 0$, we have

$$S_n(2k;x) < (-1)^{(n+1)} \psi^{(n)}(x) < S_n(2k+1;x),$$

where

$$S_n(k;x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \sum_{i=1}^{k} \frac{B_{2j}}{x^{2i+n}} \prod_{j=1}^{n-1}(2i+j).$$

Lemma 2.2 ([10,22]). For $x > 0$, we have

$$\psi'(x+1) > \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5}. \tag{2.1}$$

3. Proof of Theorem 1.1

Now we are in a position to prove our Theorem 1.1. Let

$$f(x) = \psi(x+1) - \ln x - \ln\cosh\frac{1}{\sqrt{x}} + \gamma, \quad x > 0.$$

A direct differentiation yields

$$f'(x) = \psi'(x+1) - \frac{1}{x} - \frac{e^{2/\sqrt{x}} - 1}{2\left(1+e^{2/\sqrt{x}}\right)x^{3/2}}$$

and

$$\begin{aligned} &2f'(x)\left(1+e^{2/\sqrt{x}}\right)x^{3/2} = 2\left(1+e^{2/\sqrt{x}}\right)x^{3/2}\psi'(x+1) \\ &-1+\left(1-2\sqrt{x}\right)e^{2/\sqrt{x}} - 2\sqrt{x} \\ &= \left(1+e^{2/\sqrt{x}}\right)\left[1-2\sqrt{x}+2x^{3/2}\psi'(x+1)\right] - 2 \triangleq g(x). \end{aligned}$$

By virtue of inequalities (2.1) and

$$e^{2/\sqrt{x}} > 1 + \frac{2}{\sqrt{x}} + \frac{2}{x} + \frac{4}{3x\sqrt{x}} + \frac{2}{3x^2} + \frac{4}{15x^2\sqrt{x}} + \frac{4}{45x^3}$$

we acquire

$$\begin{aligned} g(x) &> (2 + \frac{2}{\sqrt{x}} + \frac{2}{x} + \frac{4}{3x\sqrt{x}} + \frac{2}{3x^2} \\ &\quad + \frac{4}{15x^2\sqrt{x}} + \frac{4}{45x^3})[1-2\sqrt{x} \\ &\quad + 2x^{2/3}(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5})] - 2 \\ &= \frac{2}{675x^{13/2}}(-2-6x^{1/2}-15x-30x^{3/2} \\ &\quad -35x^2-15x^{5/2}+90x^{7/2}+90x^4) \\ &> \frac{2}{675x^{13/2}}(-2-6x-15x-30x^2 \\ &\quad -35x^2-15x^3+90x^3+90x^4) \\ &= \frac{1}{675x^{13/2}}[90(x-1)^4+435(x-1)^3 \\ &\quad +700(x-1)^2+434(x-1)+77] \\ &> 0 \end{aligned}$$

for $x > 1$. This implies that $f'(x) > 0$ and that $f(x)$ is increasing on $(1,\infty)$.

By the asymptotic expansion

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots$$

as $x \to \infty$ in [[1], p. 259] and the well-known formula

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{3.1}$$

we easily find

$$\psi(x+1) \sim \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \cdots$$

as $x \to \infty$. Hence, it follows that

$$\begin{aligned} f(n) &= \psi(n+1) - \ln n - \ln\cosh\frac{1}{\sqrt{n}} + \gamma \\ &= \frac{1}{2n} + O\left(\frac{1}{n}\right) - \ln\frac{e^{-1/\sqrt{n}} + e^{1/\sqrt{n}}}{2} + \gamma \\ &\to \gamma, n \to \infty. \end{aligned}$$

Taking into account that $f(x)$ is increasing on $(1,\infty)$ reveals

$$f(1) \leq f(n) < \lim_{n\to\infty} f(n) = \gamma, \quad n \in \mathbb{N}. \tag{3.2}$$

Combining (1), (3.1), and (3.2) concludes that the double inequality (1.1) holds for all $n \geq 1$ and that the bounds $\alpha = 0.5662\ldots$ and $\beta = \gamma = 0.5772\ldots$ in (1.1) are the best impossible. The proof of Theorem 1.1 is complete.

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