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Using the Matrix Summability Method to Approximate the Lip (ξ(t), p) Class Functions

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Abstract Most of the summability methods are derived from the matrix means. In this paper, author has been determined the degree of approximation of certain trigonometric functions belonging to the Lip (ξ(t), p) class by matrix method.

Keywords: *matrix means, degree of approximation, generalized Lipschitz class functions*

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1. Introduction

The Fourier series associate with f at point x of 2π periodic function in (-π, π) is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right). \tag{1.1}$$

A function $f \in \text{Lip}\,\alpha$ if

$$f(x+t) - f(x) = O\left(|t|^{\alpha}\right), \quad for \quad 0 < \alpha \le 1.$$

$$f \in Lip(\alpha, p), \quad for \quad 0 \le x \le 2\pi \text{, if}$$

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{\frac{1}{p}} = O\left(|t|^{\alpha}\right), \text{ for } 0 < \alpha \le 1$$

and an integer $p \ge 1$.

$$f \in Lip\left(\xi(t), p\right) \text{ if}$$

$$\left(\int_a^b |f(x+t) - f(x)|^p dx\right)^{\frac{1}{p}} = O(\xi(t))$$

provided $\xi(t)$ is positive increasing function.

If $\xi(t) = t^{\alpha}$ then Lip (ξ(t), p) coincide with Lip(α, p) and if $p \to \infty$ in Lip(α, p) than Lip(α, p) reduce to Lip α.

We observed that Lip α ⊆ Lip (α, p) ⊆ Lip (ξ(t), p) for $0 < \alpha \le 1$.

We define norm $\|.\|_p$ by

$$\| f \|_p = \left\{\int_0^{2\pi} |f(x)|^p dx\right\}^{\frac{1}{p}}, \; p \ge 1. \tag{1.2}$$

The degree of approximation $E_n(f)$ is given by

$$E_n(f) = \min \| t_n - f \|_p \tag{1.3}$$

where $t_n(x)$ is a trigonometric polynomial of degree n.

Let $T=(a_{n,k})$ be an infinite lower triangular matrix satisfying the condition(see, [4]) of regularity. Let $\sum_{m=0}^{\infty} u_n$ *be an* infinite series such that whose n^{th} partial sum $s_n = \sum_{k=0}^{n} u_k$.

The sequence-to-sequence transformation $t_n = \sum_{k=0}^{n} a_{n,k}\, s_k$ defines the sequence $\{t_n\}$ of lower triangular matrix means of the sequence $\{s_n\}$ generated by the sequence of coefficients $(a_{n,k})$.

The series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the sum s by lower triangular matrix method (see, [1]) if $\lim_{n\to\infty} t_n = s$.

In this paper, we use following notations.

$$\varphi(t) = f(x+t) + f(x-t) - 2f(x) \tag{1.4}$$

$$A_{n,\tau} = \sum_{k=n-\tau}^{n} a_{n,k}, \tag{1.5}$$

where $\tau = \left[\frac{1}{t}\right]$ is the greatest integer not greater than (1/t)

and

$$M_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k} \frac{\sin(k+1/2)t}{\sin(t/2)}. \tag{1.6}$$

2. Main Theorem

Chandra[5] proved a theorem on the approximation of function belonging to Lip(α,p) class by Nörlund and Riesz means. Mittal et. al. [3] extended the result of Chandra [5] by using the matrix means on same Lip(α,p) class function. In [6], Lal & Dhakal proved a theorem on approximation of a Lip α class function by matrix means.

Aim of this paper is to extend the theorems of Chandra [5], Mittal et.al. [3] and Lal & Dhakal [6] by using matrix means on Lip(ξ(t), p) class functions as following way:

Theorem. Let T=($a_{n.k}$) be an infinite regular lower triangular matrix such that the element ($a_{n.k}$) be non-negative, non-decreasing with $k \le n$. If a function $f:\Re \to \Re$ is 2π-periodic, Lebesgue integrable on [-π, π] and belonging to Lip(ξ(t), p) class then the degree of approximation of f by lower triangular matrix means $t_n(x)=\sum_{k=0}^{n} a_{n,k}s_k(x)$ of its Fourier series (1.1) satisfies, for n= 0,1,2,3…,

$$\| t_n - f \|_p = O\left((n+1)^{\frac{1}{p}}\, \xi\left(\frac{1}{n+1}\right)\right), \qquad (2.1)$$

provided ξ (t) satisfies the following conditions:

$$\left[\int_0^{\frac{1}{n+1}} \left(\frac{t\,|\varphi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = O\left(\frac{1}{n+1}\right), \qquad (2.2)$$

$$\left[\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\,|\varphi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}} = O\left((n+1)^{\delta}\right) \qquad (2.3)$$

where δ is an arbitrary number such that q(1-δ)-1 > 0, q the conjugate index of p and conditions (2.2) & (2.3) hold uniformly in x.

3. Lemmas

For the proof of the theorem following lemmas are required.

Lemma 1. M_n (t) = O (n +1), if $0 \le t \le 1/(n+1)$.

Proof. For $0 \le t \le 1/(n+1)$, $\sin(n+1)t \le (n+1)t$,

$$|M_n(t)| \le \left|\frac{1}{2}\sum_{k=0}^{n} a_{n,k}\,\frac{(2k+1)(t/2)}{t}\right|$$
$$\le \frac{(2n+1)}{4}\sum_{k=0}^{n} a_{n,k}$$
$$= O(n+1).$$

Lemma 2. If ($a_{n,k}$) is non- negative and non- decreasing with k≤n, then,

$$\left|\sum_{k=0}^{n} a_{n,k}e^{ikt}\right| = O(A_{n,\tau}), \text{ uniformly in } 0 < t \le \pi.$$

Proof: Let $\tau = \left[\frac{1}{t}\right] \le n$. Then

$$\left|\sum_{k=0}^{n} a_{n,k}e^{ikt}\right| = \left|\sum_{k=0}^{n-\tau-1} a_{n,k}e^{ikt} + \sum_{k=n-\tau}^{n} a_{n,k}e^{ikt}\right|$$
$$\le \left|\sum_{k=0}^{n-\tau-1} a_{n,k}e^{ikt}\right| + \left|\sum_{k=n-\tau}^{n} a_{n,k}e^{ikt}\right|,$$

by Abel's lemma,

$$\left|\sum_{k=0}^{n-\tau-1} a_{n,k}e^{ikt}\right| \le 2a_{n,n-\tau-1}\max_{1\le k\le n\le n-\tau-1}\left|\frac{1-e^{i(k+1)t}}{1-e^{it}}\right|$$
$$\le 4a_{n,n-\tau-1}\left|\frac{e^{it/2}}{e^{-it/2}-e^{it/2}}\right|$$
$$\le 2a_{n,n-\tau-1}\left(\frac{1}{\sin\frac{t}{2}}\right)$$
$$\le \frac{2\pi\, a_{n,n-\tau-1}}{t}$$

and

$$A_{n,\tau} = \sum_{k=n-\tau}^{n} a_{n,k} = a_{n,n-\tau} + a_{n,n-\tau++1} + \ldots + a_{n,n}$$
$$\ge a_{n,n-\tau-1} + a_{n,n-\tau-1} + \ldots + a_{n,n-\tau-1}$$
$$= (\tau+1)\,a_{n,n-\tau-1}$$
$$\ge \frac{a_{n,n-\tau-1}}{t},$$

therefore

$$\left|\sum_{k=0}^{n-\tau-1} a_{n,k}e^{ikt}\right| \le 2\pi A_{n,\tau}.$$

Also, $\left|\sum_{k=n-\tau}^{n} a_{n,k}e^{ikt}\right| \le \sum_{k=n-\tau}^{n} a_{n,k} = A_{n,\tau}.$

Thus $\left|\sum_{k=0}^{n} a_{n,k}e^{ikt}\right| \le (2\pi+1)A_{n,\tau} = O(A_{n,\tau}).$

Lemma 3. $M_n(t) = O\left(\frac{A_{n,\tau}}{t}\right)$, if $\frac{1}{n+1} < t \le \pi$.

Proof: For $\frac{1}{n+1} < t \le \pi$, $\sin(t/2) \ge (t/\pi)$, we have

$$|M_n(t)| \le \left|\frac{1}{2t}\,\mathrm{Im}\sum_{k=0}^{n} a_{n,k}\, e^{i(k+\frac{1}{2})t}\right|$$
$$\le \frac{1}{2t}\left|\sum_{k=0}^{n} a_{n,k}\, e^{ikt}\right|\left|e^{it/2}\right|$$
$$= O\left(\frac{A_{n,\tau}}{t}\right), \text{using lemma 2.}$$

4. Proof of the Theorem

Following [2], we have

$$s_k(x) - f(x) = \frac{1}{2\pi}\int_0^{\pi} \varphi(t)\frac{\sin(k+1/2)t}{\sin(t/2)}dt$$

then,

$$\sum_{k=0}^{n} a_{n,k}\left(s_{n,k}(x)-f(x)\right) = \frac{1}{2\pi}\int_0^{\pi}\varphi(t)\sum_{k=0}^{n} a_{n,k}\,\frac{\sin\left(k+1/2\right)t}{\sin(t/2)}dt$$

or

$$t_n(x)-f(x)=\int_0^{\pi}\varphi(t)M_n(t)\,dt$$

$$=\left[\int_0^{1/(n+1)}\varphi(t)\,M_n(t)\,dt+\int_{1/(n+1)}^{\pi}\varphi(t)\,M_n(t)\,dt\right] \quad (4.1)$$

$$=I_1+I_2,\text{say.}$$

For, I_1,

Applying Holder's inequality and fact that $\varphi(t)\in Lip(\xi(t),p)$, we have

$$|I_1|\le\left[\int_0^{\frac{1}{n+1}}\left(\frac{t|\varphi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t}\,(M_n)(t)\right\}^q dt\right]^{\frac{1}{q}}$$

$$=O\left(\frac{1}{n+1}\right)O\left[\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t}(n+1)\right)^q dt\right]^{\frac{1}{q}},$$

by condition (2.2) & Lemma 1.

Applying Second Mean Value Theorem for Integrals, we have

$$=O\left(\xi\left(\frac{1}{n+1}\right)\right)\left[\int_{\in}^{\frac{1}{n+1}} t^{-q}\,dt\right]^{\frac{1}{q}},$$

for some $0<\in<\dfrac{1}{n+1}$

$$=O\left(\xi\left(\frac{1}{n+1}\right)\right)\left[\left\{\frac{t^{-q+1}}{-q+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{q}} \quad (4.2)$$

$$=O\left((n+1)^{1-\frac{1}{q}}\,\xi\left(\frac{1}{n+1}\right)\right)$$

$$=O\left((n+1)^{\frac{1}{p}}\,\xi\left(\frac{1}{n+1}\right)\right)$$

$$\because\frac{1}{p}+\frac{1}{q}=1.$$

Applying Holder's inequality, condition (2.3) and Lemma 2.

$$|I_2|\le\left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}\,|\varphi(t)|}{\xi(t)}\right)^p dt\right]^{\frac{1}{p}}\left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta}}\,(M_n)(t)\right)^q dt\right]^{\frac{1}{q}}$$

$$=O\left((n+1)^{\delta}\right)O\left(\left[\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta}}\,\frac{A_{n,\tau}}{t}\right)^q dt\right]^{\frac{1}{q}}\right),$$

$$=O\left((n+1)^{\delta}\right)\left[\int_{\frac{1}{\pi}}^{n+1}\left(\frac{\xi\left(\frac{1}{u}\right)A_{n,u}}{\left(\frac{1}{u}\right)^{(1-\delta)}}\right)^q\frac{du}{u^2}\right]^{\frac{1}{q}},$$

taking $t=\dfrac{1}{u}$

$$=O\left((n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)A_{n,n}\right)\left[\int_{\frac{1}{\pi}}^{n+1}u^{q(1-\delta)-2}\,du\right]^{\frac{1}{q}}$$

$$=O\left((n+1)^{\delta}\,\xi\left(\tfrac{1}{n+1}\right)\right)\left[\left(\frac{u^{q(1-\delta)-1}}{q(1-\delta)-1}\right)_{\frac{1}{\pi}}^{n+1}\right]^{\frac{1}{q}}$$

$$=O\left((n+1)^{\delta}\,\xi\left(\tfrac{1}{n+1}\right)\right)\left((n+1)^{1-\delta-\frac{1}{q}}\right) \quad (4.3)$$

$$=O\left((n+1)^{1-\frac{1}{q}}\,\xi\left(\tfrac{1}{n+1}\right)\right)$$

$$=O\left((n+1)^{\frac{1}{p}}\,\xi\left(\tfrac{1}{n+1}\right)\right)$$

Combining the conditions (4.1) – (4.3), we have

$$\left|t_n-f(x)\right|=O\left((n+1)^{\frac{1}{p}}\,\xi\left(\tfrac{1}{n+1}\right)\right).$$

Now,

$$\left\|t_n-f\right\|_p=\left[\int_0^{2\pi}|t(x)-f(x)|^p\,dx\right]^{\frac{1}{p}}$$

$$=O\left[\int_0^{2\pi}\left\{(n+1)^{\frac{1}{p}}\,\xi\left(\tfrac{1}{n+1}\right)\right\}^p dx\right]^{\frac{1}{p}} \quad (4.4)$$

$$=O\left((n+1)^{\frac{1}{p}}\,\xi\left(\tfrac{1}{n+1}\right)\right)\left[\int_o^{2\pi}dx\right]^{\frac{1}{p}}$$

$$=O\left((n+1)^{\frac{1}{p}}\,\xi\left(\tfrac{1}{n+1}\right)\right)$$

This completes the proof of the theorem.

5. Corollaries

Following corollaries can be derived from the theorem.

Corollary 1. If ξ(t) = t^{α} then degree of approximation of a function $f \in Lip(\alpha, p)$ class is given by

$$\left\| t_n - f \right\|_p = O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{p}}} \right)$$

for $1 \leq p < \infty$, $\frac{1}{p} < \alpha \leq 1$

Corollary 2. If ξ(t) = t^{α} with p→ ∞, then degree of approximation of a function $f \in Lip\,\alpha$ class is given by

$$\left\| t_n - f \right\|_\infty = O\left(\frac{1}{(n+1)^{\alpha}} \right), \text{for } 0 < \alpha < 1.$$

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