Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 5, 189-192
Available online at http://pubs.sciepub.com/tjant/2/5/6

DOI:10.12691/tjant-2-5-6

Fibonacci Polynomials and Determinant Identities

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Abstract The Fibonacci polynomials and Lucas polynomials are famous for possessing wonderful and amazing properties and identities. In this paper, some determinant identities of Fibonacci polynomials are describe. Entries of determinants are satisfying the recurrence relations of Fibonacci polynomials and Lucas polynomials.

Keywords: *Fibonacci number, Fibonacci polynomial, Lucas polynomial, determinant*

Cite This Article: Omprakash Sikhwal, and Yashwant Vyas, “Fibonacci Polynomials and Determinant Identities.” *Turkish Journal of Analysis and Number Theory*, vol. 2, no. 6 (2014): 189-192. doi: 10.12691/tjant-2-5-6.

1. Introduction

Fibonacci numbers are a popular topic for mathematical enrichment and popularization. They are famous for a host of interesting and surprising properties and show up in text books, magazine articles, and web sites. Various sequences of polynomials by the names of Fibonacci and Lucas polynomials occur in the literature over a century. The Fibonacci polynomials and Lucas polynomials are closely related and widely investigated. Fibonacci polynomials appear in different frameworks. Fibonacci polynomials are special cases of Chebyshev polynomials and have been studied on a more advanced level by many mathematicians.

S. L. Basin [15] show that Q matrix generates a set of Fibonacci Polynomials satisfying the recurrence relation

$$f_{n+1}(x) = xf_n(x) + f_{n-1},(x), n \geq 2$$

with

$$f_0(x) = 0, f_1(x) = 1. \quad (1.1)$$

The Lucas Polynomials [1] are defined by the recurrence formula

$$l_{n+1}(x) = xl_n(x) + l_{n-1},(x), n \geq 2$$

with

$$l_0(x) = 2, l_1(x) = x. \quad (1.2)$$

Generating function of Fibonacci polynomials is

$$\sum_{n=0}^{\infty} f_n(x)t^n = t\left(1 - xt - t^2\right)^{-1}. \quad (1.3)$$

Generating function of Lucas polynomials is

$$\sum_{n=0}^{\infty} l_n(x)t^n = (2 - xt)\left(1 - xt - t^2\right)^{-1}. \quad (1.4)$$

Explicit sum formula for (1.1) is given by

$$f_n(x) = \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n-k-1}{k} x^{n-1-2k}, \quad (1.5)$$

where $\binom{n}{m}$ a binomial coefficient and [x] is define as the greatest integer less than or equal to x.

Explicit sum formula for (1.2) is given by

$$l_n(x) = \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}, \quad (1.6)$$

where $\binom{n}{m}$ a binomial coefficient and [x] is defined as the greatest integer less than or equal to x.

Determinants have played a significant part in various areas in mathematics. For instance, they are quite useful in the analysis and solution of system of linear equations. There are different perspectives on the study of determinants. One may notice several practical and effective instruments for calculating determinants in the nice survey articles [7] and [8].

Much attention has been paid to the evaluation of determinants of matrices, especially when their entries are given recursively [8].

There is a long tradition of using matrices and determinants to study Fibonacci numbers. Bicknell – Johnson and Spears [11] use elementary matrix operations and determinants to generate classes of identities for generalized Fibonacci numbers. Cahill and Narayan [12] show how Fibonacci and Lucas numbers arise as determinants of some tridiagonal matrices. A. Benjamin, T. Cameron and J. Quinn [2], provides combinatorial interpretations for Fibonacci identities using determinants. T. Koshy [16] explained two chapters on the use of matrices and determinants in Fibonacci numbers. O. Sikhwal [13] explained determinants identities of Fibonacci sequences and its generalizations.

The Fibonacci and Lucas polynomials possess many fascinating properties which have been studied in [1] to [9] and [11] to [15]. In this paper, some determinant identities of Fibonacci polynomials are describe.

2. Determinan Identities

We define a family of Fibonacci polynomial as

$$B=\left\{ f_{n+p}(x), f_{n+q}(x), f_{n+q+r}(x), f_{n+s}(x), f_{n+s+r}(x) \right\},$$

where n and p are non negative integers, q and s are positive integers with $0 \le p<q$, q+1<s r=1.

Assume $f_{n+p}(x)=a$, $f_{n+q}(x)=b$, then by (1.1) $f_{n+q+r}(x)=a+bx$, $f_{n+q+r}(x)=xf_{n+q}+f_{n+p}$, $f_{n+s}(x)=xf_{n+q+r}+f_{n+q}$, $f_{n+s+r}(x)=xf_{n+s}(x)+f_{n+q+r}$.

Theorem 1: If n and p are non-negative integers, q is positive integer with $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} f_{n+p}(x) & f_{n+q}(x) & f_{n+q+r}(x) \\ f_{n+q+r}(x) & f_{n+p}(x) & f_{n+q}(x) \\ f_{n+q}(x) & f_{n+q+r}(x) & f_{n+p}(x) \end{vmatrix} = f^3_{n+p}(x)+f^3_{n+q}(x)+f^3_{n+q+r}(x)-3f_{n+p}(x)f_{n+q}(x)f_{n+q+r}(x)$$

Proof:
Let

$$\Delta=\begin{vmatrix} f_{n+p}(x) & f_{n+q}(x) & f_{n+q+r}(x) \\ f_{n+q+r}(x) & f_{n+p}(x) & f_{n+q}(x) \\ f_{n+q}(x) & f_{n+q+r}(x) & f_{n+p}(x) \end{vmatrix} \quad (2.1)$$

Assume $f_{n+p}(x)=a$, $f_{n+q}(x)=b$, then by (1.1) $f_{n+q+r}(x)=a+bx$

Now

$$\Delta=\begin{vmatrix} a & b & a+bx \\ a+bx & a & b \\ b & a+bx & a \end{vmatrix} \quad (2.2)$$

Applying $R_1+R_2 \to R_1$,

$$\Delta=\begin{vmatrix} 2a+bx & a+b & a+b+bx \\ a+bx & a & b \\ b & a+bx & a \end{vmatrix} \quad (2.3)$$

Applying $C_1-C_2 \to C_1$,

$$\Delta=\begin{vmatrix} a+bx-b & a+b & a+b+bx \\ bx & a & b \\ b-a-bx & a+bx & a \end{vmatrix} \quad (2.4)$$

Applying $R_1+R_3 \to R_1$,

$$\Delta=\begin{vmatrix} 0 & 2a+b+bx & 2a+b+bx \\ bx & a & b \\ b-a-bx & a+bx & a \end{vmatrix} \quad (2.5)$$

Applying $C_2-C_3 \to C_2$,

$$\Delta=\begin{vmatrix} 0 & 0 & 2a+b+bx \\ bx & a-b & b \\ b-a-bx & bx & a \end{vmatrix}. \quad (2.6)$$

Expand along first row, we get

$$\Delta=(a+bx)^3+a^3+b^3-3ab(a+bx).$$

Put $f_{n+p}(x)=a$, $f_{n+q}(x)=b$, $f_{n+q+r}(x)=a+bx$, we get

$$\begin{vmatrix} f_{n+p}(x) & f_{n+q}(x) & f_{n+q+r}(x) \\ f_{n+q+r}(x) & f_{n+p}(x) & f_{n+q}(x) \\ f_{n+q}(x) & f_{n+q+r}(x) & f_{n+p}(x) \end{vmatrix} = f^3_{n+p}(x)+f^3_{n+q}(x)+f^3_{n+q+r}(x) -3f_{n+p}(x)f_{n+q}(x)f_{n+q+r}(x). \quad (2.7)$$

Corollary 1.1: If we put x = 1 in above result, for $0 \le p<q$, r=1, we get

$$\begin{vmatrix} F_{n+p} & F_{n+q} & F_{n+q+r} \\ F_{n+q+r} & F_{n+p} & F_{n+q} \\ F_{n+q} & F_{n+q+r} & F_{n+p} \end{vmatrix} = F^3_{n+p}+F^3_{n+q}+F^3_{n+q+r}-3F_{n+p}F_{n+q}F_{n+q+r} \quad (2.8)$$
$$=2\left(F^3_{n+p}+F^3_{n+q}\right).$$

It can be proved easily.

Theorem 2: If n and p are non-negative integers, q is positive integer with $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} f_{n+p}(x) & l_{n+p}(x) & 1 \\ f_{n+q}(x) & l_{n+q}(x) & 1 \\ f_{n+q+r}(x) & l_{n+q+r}(x) & 0 \end{vmatrix} = (x+1)\left[f_{n+q}(x)l_{n+p}(x)-f_{n+p}(x)l_{n+q}(x) \right] \quad (2.9)$$
$$=\begin{cases} 2(1+x), \; if \; n=0,2,4... \\ -2(1+x), \; if \; n=1,3,5,... \end{cases}$$

Proof: Let $\Delta=\begin{vmatrix} f_{n+p}(x) & l_{n+p}(x) & 1 \\ f_{n+q}(x) & l_{n+q}(x) & 1 \\ f_{n+q+r}(x) & l_{n+q+r}(x) & 0 \end{vmatrix}$

Assume $f_{n+p}(x)=a$, $f_{n+q}(x)=b$, then by (1.1) $f_{n+q+r}(x)=a+bx$ and $l_{n+p}(x)=c$, $l_{n+q}(x)=d$, then by (1.2), $l_{n+q+r}(x)=c+dx$.

Now

$$\Delta=\begin{vmatrix} a & c & 1 \\ b & d & 1 \\ a+bx & c+dx & 0 \end{vmatrix} \quad (2.10)$$

Applying $R_1-R_2\to R_1$,

$$\Delta=\begin{vmatrix} a-b & c-d & 0 \\ b & d & 1 \\ a+bx & c+dx & 0 \end{vmatrix} \quad (2.11)$$

Interchanging C_1 and C_3,

$$\Delta=(-1)\begin{vmatrix} 0 & c-d & a-b \\ 1 & d & b \\ 0 & c+dx & a+bx \end{vmatrix} \quad (2.12)$$

Expand it, we get

$$\Delta=(x+1)[bc-ad] \quad (2.13)$$

Assume $f_{n+p}(x)=a,\quad f_{n+q}(x)=b,$ then by (1.1) $f_{n+q+r}(x)=a+bx$ and then by (1.1) $l_{n+p}(x)=c, l_{n+q}(x)=d\ \ l_{n+q+r}(x)=c+dx$, we get

$$\begin{vmatrix} f_{n+p}(x) & l_{n+p}(x) & 1 \\ f_{n+q}(x) & l_{n+q}(x) & 1 \\ f_{n+q+r}(x) & l_{n+q+r}(x) & 0 \end{vmatrix}$$

$$=(x+1)\left[f_{n+q}(x)l_{n+p}(x)-f_{n+p}(x)l_{n+q}(x)\right]$$

$$=\begin{cases} 2(1+x),\ if\ \ n=0,2,4... \\ -2(1+x),\ if\ \ n=1,3,5,... \end{cases}$$

Corollary 2.1: If we put x = 1 in above result, we get

$$\begin{vmatrix} F_{n+p} & L_{n+p} & 1 \\ F_{n+q} & L_{n+p} & 1 \\ F_{n+q+r} & L_{n+q+r} & 0 \end{vmatrix} \quad (2.14)$$

$$=2\left[F_{n+q}L_{n+p}-F_{n+p}L_{n+q}\right]=\begin{cases} 4, n=0,2,4,... \\ -4, n=1,3,5,... \end{cases}$$

It can be proved easily.

Theorem 3: If n and p are non-negative integers, q and s are positive integers with $0\le p<q$, $q+1<s$ $r=1$, Prove that

$$\begin{vmatrix} f_{n+p}(x) & f_{n+q}(x) & f_{n+q+r}(x) \\ f_{n+q}(x) & f_{n+q+r}(x) & f_{n+s}(x) \\ f_{n+q+r}(x) & f_{n+s}(x) & f_{n+s+r}(x) \end{vmatrix}=0$$

Proof: Assume $f_{n+p}(x)=a,\quad f_{n+q}(x)=b,$ then by (1.1) $f_{n+q+r}(x)=a+bx$ and $f_{n+q+r}(x)=xf_{n+q}+f_{n+p}$, $f_{n+s}(x)=xf_{n+q+r}+f_{n+q}$, $f_{n+s+r}(x)=xf_{n+s}(x)+f_{n+q+r}$.

Let

$$\Delta=\begin{vmatrix} f_{n+p}(x) & f_{n+q}(x) & f_{n+q+r}(x) \\ f_{n+q}(x) & f_{n+q+r}(x) & f_{n+s}(x) \\ f_{n+q+r}(x) & f_{n+s}(x) & f_{n+s+r}(x) \end{vmatrix} \quad (2.15)$$

Applying $C_1+xC_2\to C_1$,

$$\Delta=\begin{vmatrix} f_{n+q+r}(x) & f_{n+q}(x) & f_{n+q+r}(x) \\ f_{n+s}(x) & f_{n+q+r}(x) & f_{n+s}(x) \\ f_{n+s+r}(x) & f_{n+s}(x) & f_{n+s+r}(x) \end{vmatrix} \quad (2.16)$$

Here two columns are identical, we get

$$\Delta=\begin{vmatrix} f_{n+p}(x) & f_{n+q}(x) & f_{n+q+r}(x) \\ f_{n+q}(x) & f_{n+q+r}(x) & f_{n+s}(x) \\ f_{n+q+r}(x) & f_{n+s}(x) & f_{n+s+r}(x) \end{vmatrix}=0 \quad (2.17)$$

Corollary 3.1: If we put x = 1 in above result, we get

$$\begin{vmatrix} F_{n+p} & F_{n+q} & F_{n+q+r} \\ F_{n+q} & F_{n+q+r} & F_{n+s} \\ F_{n+q+r} & F_{n+s} & F_{n+s+r} \end{vmatrix}=0 \quad (2.18)$$

It can be proved easily.

Theorem 4: If n and p are non-negative integers, q is positive integer with $0\le p<q$, $r=1$, Prove that

$$\begin{vmatrix} \left(f_{n+p}(x)+f_{n+q}(x)\right)^2 & f_{n+p}(x)f_{n+q+r}(x) & f_{n+q}(x)f_{n+q+r}(x) \\ f_{n+p}(x)f_{n+q+r}(x) & \left(f_{n+q}(x)+f_{n+q+r}(x)\right)^2 & f_{n+p}(x)f_{n+q}(x) \\ f_{n+q}(x)f_{n+q+r}(x) & f_{n+p}(x)f_{n+q}(x) & \left(f_{n+q+r}(x)+f_{n+p}(x)\right)^2 \end{vmatrix} \quad (2.19)$$

$$=2f_{n+p}(x)f_{n+q}(x)f_{n+q+r}(x)\left(f_{n+p}(x)+f_{n+q}(x)+f_{n+q+r}(x)\right)^3$$

It can be proved same as Theorem 1.

Theorem 5: If n and p are non-negative integers, q is positive integer with $0\le p<q$, $r=1$, Prove that

$$\begin{vmatrix} f_{n+p}(x) & f_{n+q}(x) & f_{n+q+r}(x) \\ f_{n+p}(x)-f_{n+q}(x) & f_{n+q}(x)-f_{n+q+r}(x) & f_{n+q+r}(x)-f_{n+p}(x) \\ f_{n+q}(x)+f_{n+q+r}(x) & f_{n+p}(x)+f_{n+q+2}(x) & f_{n+p}(x)+f_{n+q}(x) \end{vmatrix}$$

$$=f_{n+p}{}^3(x)+f_{n+q}{}^3(x)+f_{n+q+r}{}^3(x) \quad (2.20)$$

$$-3f_{n+p}(x)f_{n+q}(x)f_{n+q+r}(x)$$

It can be proved same as Theorem 1.

Theorem 6: If n and p are non-negative integers, q and s are positive integers with $0\le p<q$, $q+1<s$ $r=1$, Prove that

$$\begin{vmatrix} f_{n+s}(x) & f_{n+q+r}(x) & f_{n+q}(x) & f_{n+p}(x) \\ f_{n+q+r}(x) & f_{n+s}(x) & f_{n+p}(x) & f_{n+q}(x) \\ f_{n+q}(x) & f_{n+p}(x) & f_{n+s}(x) & f_{n+q+r}(x) \\ f_{n+p}(x) & f_{n+q}(x) & f_{n+q+r}(x) & f_{n+s}(x) \end{vmatrix}$$

$$=x^2\left[f_{n+p}(x)+f_{n+q}(x)+f_{n+q+r}(x)+f_{n+s}(x)\right] \quad (2.21)$$

$$\left[f_{n+q+r}{}^2(x)-f_{n+q}{}^2(x)\right]$$

Proof: Assume $f_{n+p}(x)=a,\quad f_{n+q}(x)=b,$ then by (1.1) $f_{n+q+r}(x)=a+bx$ and $f_{n+q+r}(x)=xf_{n+q}+f_{n+p}$, $f_{n+s}(x)=xf_{n+q+r}+f_{n+q}$,

It can be proved same as Theorem 1.

Corollary 6.1: If we put x = 1 in above result, we get

$$\begin{vmatrix} F_{n+s}(x) & F_{n+q+r}(x) & F_{n+q}(x) & F_{n+p}(x) \\ F_{n+q+r}(x) & F_{n+s}(x) & F_{n+p}(x) & F_{n+q}(x) \\ F_{n+q}(x) & F_{n+p}(x) & F_{n+s}(x) & F_{n+q+r}(x) \\ F_{n+p}(x) & F_{n+q}(x) & F_{n+q+r}(x) & F_{n+s}(x) \end{vmatrix} \quad (2.22)$$

$$= \left[F_{n+p}(x) + F_{n+q}(x) + F_{n+q+r}(x) + F_{n+s}(x)\right]$$

$$F_{n+p}(x)\left[4F_{n+q}{}^2(x) - F_{n+p}{}^2(x)\right]$$

$$= F_{2(n+p)+6}(x).F_{2(n+p)}(x).$$

It can be proved easily.

Theorem 7: If n and p are non-negative integers, q is positive integer with $0 \le p < q$, $r=1$ and $\alpha = f_{n+p}(x)$, $\beta = f_{n+q}(x)$, $\gamma = f_{n+q+r}(x)$, Prove that

$$\begin{vmatrix} \alpha\gamma+\beta^2 & \alpha^2 & \beta^2 & \alpha\gamma^2 & -\left(\alpha\gamma+\beta^2\right) \\ \alpha\gamma+\beta^2 & \alpha\gamma & -\alpha\gamma & \alpha\gamma^2 & \alpha\gamma+\beta^2 \\ 0 & 2\alpha\gamma & 2\alpha\gamma & -2\alpha\gamma^2 & 0 \\ \alpha\gamma+\beta^2 & -\alpha\gamma & \alpha\gamma & -\alpha\gamma^2 & \alpha\gamma+\beta^2 \\ -\left(\alpha\gamma+\beta^2\right) & \alpha^2 & \beta^2 & \alpha\gamma^2 & \alpha\gamma+\beta^2 \end{vmatrix} \quad (2.23)$$

$$= -32\alpha^2\gamma^3\left(\alpha\gamma+\beta^2\right)$$

$$= -32 f_{n+p}{}^2(x) f_{n+q+r}{}^3(x)$$

$$\left[f_{n+p}(x) f_{n+q+r}(x) + f_{n+q}{}^2(x)\right]^3.$$

Proof: Assume $f_{n+p}(x) = a,\quad f_{n+q}(x) = b,$ then by (1.1) $f_{n+q+r}(x) = a + bx$.

It can be proved same as Theorem 1.

3. Conclusion

This paper describes determinant identities of Fibonacci polynomials. Determinants identities included various pattern of Fibonacci polynomials. Few results connected with Lucas polynomials. More identities can be developed with generalized polynomials and other classical polynomials.

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