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Some New Ostrowski Type Inequalities for Co-Ordinated Convex Functions

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Abstract In this paper, we obtain new identity for function of two variables and apply them to give new Ostrowski type integral inequality for double integrals involving functions whose derivatives are co-ordinates convex function on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b,\ c<d$.

Keywords: *Ostrowski type inequalities, coordinated convex functions, Hölder's inequality*

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1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is baunded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)|<\infty$. Then, we have the inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty , \tag{1.1}$$

for all $x\in[a,b]$ (see, [13]). The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality*. For some results which generalize, improve and extend the inequality (1.1) see ([5,6,7,14,15,16,17]) and the references therein.

Let us consider now a bidimensional interval $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$, a mapping $f:\Delta\to\mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le \lambda f(x,y)+(1-\lambda)f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$. The mapping f is said to be concave on the co-ordinates on if the above inequality holds in reverse direction, for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

A formal definition for coordinated convex function may be stated as follows:

Definition 1. A function $f:\Delta\to\mathbb{R}$ will be coordinated convex on Δ, for all $t,s\in[0,1]$ and $(x,y),(u,v)\in\Delta$, if the following inequality holds:

$$\begin{aligned} &f(tx+(1-t)y,su+(1-s)v)\\ &\le tsf(x,u)+s(1-t)f(y,u)\\ &\quad +t(1-s)f(x,v)+(1-t)(1-s)f(y,v). \end{aligned}$$

Clearly, every convex function is coordinated convex. Furthermore, there exist coordinated convex function which is not convex, (see, [3]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1,2,3,4,8-12,18,19]).

Also, in [3], Dragomir establish the following Hermite-Hadamard's type inequality for coordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1. Suppose that $f:\Delta\to\mathbb{R}$ is coordinated convex on Δ. Then one has the inequalities:

$$\begin{aligned} &f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\ &\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\ &\le\frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right.\\ &\quad\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right]\\ &\le\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned} \tag{1.2}$$

The above inequalities are sharp.

In a recent paper [5], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals:

Theorem 2. Let $f:\Delta\to\mathbb{R}$ be continuous on $[a,b]\times[c,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,

$$\left\|f''_{x,y}\right\|_\infty=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|<\infty.$$

Then, we have the inequality:

$$\begin{aligned}&\left|\int_a^b\int_c^d f(s,t)dtds-(d-c)(b-a)f(x,y)\right.\\&\left.-\left[(b-a)\int_c^d f(x,t)dt+(d-c)\int_a^b f(s,y)ds\right]\right|\\&\le\left[\frac{1}{4}(b-a)^2+(x-\frac{a+b}{2})^2\right]\\&\times\left[\frac{1}{4}(d-c)^2+(y-\frac{d+c}{2})^2\right]\left\|f''_{x,y}\right\|_\infty\end{aligned}\tag{1.3}$$

for all $(x,y)\in[a,b]\times[c,d]$.

The main aim of this paper is to establish some new Ostrowski type inequalities for double integrals involving functions whose derivatives are co-ordinates convex function on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b,\ c<d$.

2. Main Results

To establish our main results, we need the following identity:

Lemma 1. Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ. If $f_{\lambda\alpha}=\frac{\partial^2 f}{\partial\lambda\partial\alpha}\in L_1(\Delta)$, then for any $(x,y)\in\Delta$, we have the equality:

$$\begin{aligned}f(x,y)=&\frac{1}{d-c}\int_c^d f(x,s)ds+\frac{1}{b-a}\int_a^b f(t,y)dt\\&-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\\&+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d(x-t)(y-s)\\&\times\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)d\alpha d\lambda dsdt.\end{aligned}\tag{2.1}$$

Proof For any $t,x\in[a,b]$ and $y,s\in[c,d], t\neq x$, $y\neq s$, we have

$$\begin{aligned}\int_t^x\int_s^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma&=\int_t^x\left[f_\sigma(\sigma,y)-f_\sigma(\sigma,s)\right]d\sigma\\&=\left[f(\sigma,y)-f(\sigma,s)\right]_t^x\\&=f(x,y)-f(x,s)-f(t,y)+f(t,s)\end{aligned}$$

and

$$f(x,y)=f(x,s)+f(t,y)-f(t,s)+\int_t^x\int_s^y f_{\sigma\tau}(\sigma,\tau)d\tau d\sigma.$$

For $\sigma=\lambda x+(1-\lambda)t$ and $\tau=\alpha y+(1-\alpha)s$, we obtain

$$\begin{aligned}f(x,y)=&f(x,s)+f(t,y)-f(t,s)+(x-t)(y-s)\\&\times\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)d\alpha d\lambda.\end{aligned}\tag{2.2}$$

By integrating (2.2) with respect to t,s on Δ and divide by $(b-a)(d-c)$, we get the desired equality (2.1).

Theorem 3. Let $f:\Delta\to\mathbb{R}$ be a partial differentiable mapping on Δ and $\left|\frac{\partial^2 f}{\partial\lambda\partial\alpha}\right|=\left|f_{\lambda\alpha}\right|$ is co-ordinates convex function on Δ.

(i) If $f_{\lambda\alpha}\in L_\infty(\Delta)$ then for any $(x,y)\in\Delta$,

$$\begin{aligned}&\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\&\le\frac{(b-a)(d-c)}{4}\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left\|f_{\lambda\alpha}(x,.)\right\|_\infty\\+\left\|f_{\lambda\alpha}(.,y)\right\|_\infty+\left\|f_{\lambda\alpha}\right\|_\infty\end{array}\right]\\&\times\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right].\end{aligned}$$

(ii) If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ then for any $(x,y)\in\Delta$,

$$\begin{aligned}&\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|\\&\le\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\\&\times\left[\left(\frac{d-y}{d-c}\right)^{q+1}+\left(\frac{y-c}{d-c}\right)^{q+1}\right]^{\frac{1}{q}}\\&\times\left\|\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|^p+\left|f_{\lambda\alpha}(x,.)\right|^p\\+\left|f_{\lambda\alpha}(.,y)\right|^p+\left|f_{\lambda\alpha}\right|^p\end{array}\right\|_p.\end{aligned}$$

(iii) If $f_{\lambda\alpha}\in L_1(\Delta)$ then for any $(x,y)\in\Delta$,

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]$$
$$\times\Big[(b-a)(d-c)\left|f_{\lambda\alpha}(x,y)\right|+(b-a)\left\|f_{\lambda\alpha}(x,.)\right\|_1+(d-c)\left\|f_{\lambda\alpha}(.,y)\right\|_1+\left\|f_{\lambda\alpha}\right\|_1\Big]$$

Proof (i). Using (2.1), convexity of $\left|f_{\lambda\alpha}\right|$ and taking the modulus, it follows that

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s|\times\int_0^1\int_0^1\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|d\alpha d\lambda dsdt$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\int_0^1\int_0^1|x-t||y-s|\times\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|d\alpha d\lambda dsdt$$
$$\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\int_0^1\int_0^1|x-t||y-s|\times\left[\begin{array}{l}\lambda\alpha\left|f_{\lambda\alpha}(x,y)\right|+\lambda(1-\alpha)\left|f_{\lambda\alpha}(x,s)\right|\\+(1-\lambda)\alpha\left|f_{\lambda\alpha}(t,y)\right|\\+(1-\lambda)(1-\alpha)\left|f_{\lambda\alpha}(t,s)\right|\end{array}\right]d\alpha d\lambda dsdt$$
$$=\frac{1}{4(b-a)(d-c)}\times\int_a^b\int_c^d|x-t||y-s|\times\left[\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,s)\right|+\left|f_{\lambda\alpha}(t,y)\right|+\left|f_{\lambda\alpha}(t,s)\right|\right]dsdt$$

Since $f_{\lambda\alpha}\in L_\infty(\Delta)$, we get

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s|dsdt\times ess\sup_{(t,s)\in\Delta}\left\{\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,s)\right|\\+\left|f_{\lambda\alpha}(t,y)\right|+\left|f_{\lambda\alpha}(t,s)\right|\end{array}\right\}$$
$$=\frac{1}{4(b-a)(d-c)}\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left\|f_{\lambda\alpha}(x,.)\right\|_\infty\\+\left\|f_{\lambda\alpha}(.,y)\right\|_\infty+\left\|f_{\lambda\alpha}\right\|_\infty\end{array}\right]\times\left(\int_a^b|x-t|dt\right)\left(\int_c^d|y-s|ds\right)$$
$$=\frac{1}{4(b-a)(d-c)}\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left\|f_{\lambda\alpha}(x,.)\right\|_\infty\\+\left\|f_{\lambda\alpha}(.,y)\right\|_\infty+\left\|f_{\lambda\alpha}\right\|_\infty\end{array}\right]\times\left(\frac{(x-a)^2+(b-x)^2}{2}\right)\left(\frac{(y-c)^2+(d-y)^2}{2}\right)$$
$$=\frac{(b-a)(d-c)}{4}\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left\|f_{\lambda\alpha}(x,.)\right\|_\infty\\+\left\|f_{\lambda\alpha}(.,y)\right\|_\infty+\left\|f_{\lambda\alpha}\right\|_\infty\end{array}\right]\times\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right].$$

(ii). As above, we can write

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s|\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,s)\right|\\+\left|f_{\lambda\alpha}(t,y)\right|+\left|f_{\lambda\alpha}(t,s)\right|\end{array}\right]dsdt.$$

Using Hölder's ineguality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4(b-a)(d-c)}\left(\int_a^b\int_c^d|x-t|^q|y-s|^q dsdt\right)^{\frac{1}{q}}\times\left(\int_a^b\int_c^d\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,s)\right|\\+\left|f_{\lambda\alpha}(t,y)\right|+\left|f_{\lambda\alpha}(t,s)\right|\end{array}\right]^p dsdt\right)^{\frac{1}{p}}$$
$$=\frac{1}{4(b-a)(d-c)}\left(\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right)^{\frac{1}{q}}\times\left(\frac{(y-c)^{q+1}+(d-y)^{q+1}}{q+1}\right)^{\frac{1}{q}}$$
$$\times\left\|\left|f_{\lambda\alpha}(x,y)\right|+\left|f_{\lambda\alpha}(x,.)\right|+\left|f_{\lambda\alpha}(.,y)\right|+\left|f_{\lambda\alpha}\right|\right\|_p.$$

(iii). As above, we obtain the following inequality

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4(b-a)(d-c)}\times\int_a^b\int_c^d|x-t||y-s|\times\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|\\+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\end{array}\right]dsdt$$

Using convexity of $|f_{\lambda\alpha}|$, we obtain

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{4(b-a)(d-c)}\sup_{t\in[a,b]}|x-t|\sup_{s\in[c,d]}|y-s|\times\int_a^b\int_c^d\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|\\+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\end{array}\right]dsdt$$
$$=\frac{1}{4(b-a)(d-c)}\max\{x-a,b-x\}\max\{y-c,d-y\}\times\left[(b-a)(d-c)|f_{\lambda\alpha}(x,y)|+(b-a)\|f_{\lambda\alpha}(x,.)\|_1+(d-c)\|f_{\lambda\alpha}(.,y)\|_1+\|f_{\lambda\alpha}\|_1\right]$$
$$=\frac{1}{4}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]\times\left[\begin{array}{l}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|\\+(b-a)\|f_{\lambda\alpha}(x,.)\|_1\\+(d-c)\|f_{\lambda\alpha}(.,y)\|_1+\|f_{\lambda\alpha}\|_1\end{array}\right]$$

This completes the proof.

Corollary 1. With the assumptions of Theorem 3 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have the inequality

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{(b-a)(d-c)}{64}\left[\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_\infty+\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_\infty+\|f_{\lambda\alpha}\|_\infty\right],$$

provided $f_{\lambda\alpha}\in L_\infty(\Delta)$.

If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{16(q+1)^{\frac{2}{q}}}\left\|\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^p+\left|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right|^p+\left|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right|^p+|f_{\lambda\alpha}|^p\right\|_p.$$

If $f_{\lambda\alpha}\in L_1(\Delta)$, then

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{1}{16}\left[\begin{array}{l}(b-a)(d-c)\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\+(b-a)\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_1\\+(d-c)\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_1+\|f_{\lambda\alpha}\|_1\end{array}\right].$$

Theorem 4. Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ and $\left|\frac{\partial^2 f}{\partial\lambda\partial\alpha}\right|^p=|f_{\lambda\alpha}|^p$, $p>1$ is co-ordinates convex function on Δ.

(i) If $f_{\lambda\alpha}\in L_\infty(\Delta)$ then for any $(x,y)\in\Delta$,

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\le\frac{(b-a)(d-c)}{4^{\frac{1}{p}}}\times\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_\infty^p\\+\|f_{\lambda\alpha}(.,y)\|_\infty^p+\|f_{\lambda\alpha}\|_\infty^p\end{array}\right]^{\frac{1}{p}}\times\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right].$$

(ii) If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ then for any $(x,y)\in\Delta$,

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}(q+1)^{\frac{2}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}$$
$$x\left[\left(\frac{d-y}{d-c}\right)^{q+1}+\left(\frac{y-c}{d-c}\right)^{q+1}\right]^{\frac{1}{q}}$$
$$\times\left[\begin{array}{l}(b-a)(d-c)\left|f_{\lambda\alpha}(x,y)\right|^p+\left\|f_{\lambda\alpha}(x,.)\right\|_p^p\\+\left\|f_{\lambda\alpha}(.,y)\right\|_p^p+\left\|f_{\lambda\alpha}\right\|_p^p\end{array}\right]^{\frac{1}{p}}.$$

(iii) If $f_{\lambda\alpha}\in L_p(\Delta)$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ then for any $(x,y)\in\Delta$,

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]$$
$$\times\left[\begin{array}{l}(b-a)(d-c)\left|f_{\lambda\alpha}(x,y)\right|^p\\+\left\|f_{\lambda\alpha}(x,.)\right\|_p^p+\left\|f_{\lambda\alpha}(.,y)\right\|_p^p+\left\|f_{\lambda\alpha}\right\|_p^p\end{array}\right]^{\frac{1}{p}}.$$

Proof As in the proof of Theorem 3, we can write

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dtt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsd\right|$$
$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s|$$
$$\int_0^1\int_0^1\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|d\alpha d\lambda dsdt$$

for any $(x,y)\in\Delta$. From Hölder's inequality, we get

$$\int_0^1\int_0^1\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|d\alpha d\lambda$$
$$\leq\left(\int_0^1\int_0^1 1^q d\alpha d\lambda\right)^{\frac{1}{q}}\left(\int_0^1\int_0^1\left|\begin{array}{l}f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y\\+(1-\alpha)s)\end{array}\right|^p d\alpha d\lambda\right)^{\frac{1}{p}}$$

for any $(x,y)\in\Delta$.

Since $\left|f_{\lambda\alpha}\right|^p$ is a co-ordinates convex function on Δ, we get

$$\left(\int_0^1\int_0^1\left|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\right|^p d\alpha d\lambda\right)^{\frac{1}{p}}$$
$$\leq\left(\frac{1}{4}\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|^p+\left|f_{\lambda\alpha}(x,s)\right|^p\\+\left|f_{\lambda\alpha}(t,y)\right|^p+\left|f_{\lambda\alpha}(t,s)\right|^p\end{array}\right]\right)^{\frac{1}{p}}$$

for any $(x,y)\in\Delta$. Therefore

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s| \tag{2.3}$$
$$\times\left(\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|^p+\left|f_{\lambda\alpha}(x,s)\right|^p\\+\left|f_{\lambda\alpha}(t,y)\right|^p+\left|f_{\lambda\alpha}(t,s)\right|^p\end{array}\right]\right)^{\frac{1}{p}}dsdt.$$

(i). Now, if $f_{\lambda\alpha}\in L_\infty(\Delta)$ then

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt\right.$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$
$$\leq\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}$$
$$\times\sup_{(t,s)\in\Delta}\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|^p+\left|f_{\lambda\alpha}(x,s)\right|^p\\+\left|f_{\lambda\alpha}(t,y)\right|^p+\left|f_{\lambda\alpha}(t,s)\right|^p\end{array}\right]^{\frac{1}{p}}$$
$$\times\int_a^b\int_c^d|x-t||y-s|dsdt$$
$$=\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}$$
$$\times\left[\begin{array}{l}\left|f_{\lambda\alpha}(x,y)\right|^p+\left\|f_{\lambda\alpha}(x,.)\right\|_\infty^p\\+\left\|f_{\lambda\alpha}(.,y)\right\|_\infty^p+\left\|f_{\lambda\alpha}\right\|_\infty^p\end{array}\right]^{\frac{1}{p}}$$
$$\times\left(\frac{(x-a)^2+(b-x)^2}{2}\right)\left(\frac{(y-c)^2+(d-y)^2}{2}\right)$$

for any $(x,y)\in\Delta$.

(ii). If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, by using Hölder's inequality in (2.3), we have

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\le\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\left(\int_a^b\int_c^d|x-t|^q|y-s|^q\,dsdt\right)^{\frac{1}{q}}$$

$$\times\left(\int_a^b\int_c^d\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|^p+|f_{\lambda\alpha}(x,s)|^p\\+|f_{\lambda\alpha}(t,y)|^p+|f_{\lambda\alpha}(t,s)|^p\end{array}\right]dsdt\right)^{\frac{1}{p}}$$

$$\le\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\left(\frac{(x-a)^2+(b-x)^2}{2}\right)\left(\frac{(y-c)^2+(d-y)^2}{2}\right)$$

$$\times\left[\begin{array}{l}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_p^p\\+\|f_{\lambda\alpha}(.,y)\|_p^p+\|f_{\lambda\alpha}\|_p^p\end{array}\right]^{\frac{1}{p}}$$

$$=\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}(q+1)^{\frac{2}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1}+\left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}$$

$$\times\left[\left(\frac{d-y}{d-c}\right)^{q+1}+\left(\frac{y-c}{d-c}\right)^{q+1}\right]^{\frac{1}{q}}$$

$$\times\left[\begin{array}{l}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_p^p\\+\|f_{\lambda\alpha}(.,y)\|_p^p+\|f_{\lambda\alpha}\|_p^p\end{array}\right]^{\frac{1}{p}}$$

for any $(x,y)\in\Delta.$

(iii) If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1,$ $\frac{1}{p}+\frac{1}{q}=1$, then, by Hölder's inequality, we have

$$\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\le\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\int_a^b\int_c^d|x-t||y-s|\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|^p+|f_{\lambda\alpha}(x,s)|^p\\+|f_{\lambda\alpha}(t,y)|^p+|f_{\lambda\alpha}(t,s)|^p\end{array}\right]^{\frac{1}{p}}dsdt$$

$$=\frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\sup_{t\in[a,b]}|x-t|\sup_{s\in[c,d]}|y-s|\left(\int_a^b\int_c^d 1^q\,dsdt\right)^{\frac{1}{q}}$$

$$\times\left(\int_a^b\int_c^d\left[\begin{array}{l}|f_{\lambda\alpha}(x,y)|^p+|f_{\lambda\alpha}(x,s)|^p\\+|f_{\lambda\alpha}(t,y)|^p+|f_{\lambda\alpha}(t,s)|^p\end{array}\right]dsdt\right)^{\frac{1}{p}}$$

$$\le\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}(b-a)(d-c)}\max\{x-a,b-x\}\max\{y-c,d-y\}$$

$$\times\left[\begin{array}{l}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_p^p\\+\|f_{\lambda\alpha}(.,y)\|_p^p+\|f_{\lambda\alpha}\|_p^p\end{array}\right]^{\frac{1}{p}}$$

$$=\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]$$

$$\times\left[\begin{array}{l}(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p+\|f_{\lambda\alpha}(x,.)\|_p^p\\+\|f_{\lambda\alpha}(.,y)\|_p^p+\|f_{\lambda\alpha}\|_p^p\end{array}\right]^{\frac{1}{p}}.$$

Corollary 2. With the assumptions of Theorem 4 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$, we have the inequality

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\le\frac{(b-a)(d-c)}{4^{2+\frac{1}{p}}}\left[\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^p+\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_\infty^p+\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_\infty^p+\|f_{\lambda\alpha}\|_\infty^p\right]^{\frac{1}{p}},$$

where $f_{\lambda\alpha}\in L_\infty(\Delta)$.

If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1,$ $\frac{1}{p}+\frac{1}{q}=1$ then we have

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right|$$

$$\le\frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{1+\frac{1}{p}}(q+1)^{\frac{2}{q}}}\left[(b-a)(d-c)\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^p+\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_p^p+\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_p^p+\|f_{\lambda\alpha}\|_p^p\right]^{\frac{1}{p}}.$$

If $f_{\lambda\alpha}\in L_p(\Delta)$, $p>1,\frac{1}{p}+\frac{1}{q}=1,$ then we have,

$$\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds \right.$$

$$\left. -\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right|$$

$$\leq \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{1+\frac{1}{p}}}\left[(b-a)(d-c)\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^p\right.$$

$$\left.+\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_p^p+\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_p^p+\left\|f_{\lambda\alpha}\right\|_p^p\right]^{\frac{1}{p}}.$$

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