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On the Simpson's Inequality for Convex Functions on the Co-Ordinates

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Abstract In this paper, a new lemma is proved and inequalities of Simpson type are established for convex functions on the co-ordinates and bounded functions.

Keywords: *Simpson's inequality, co-ordinates, convex functions, bounded functions*

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1. Introduction

The following inequality is well-known in the literature as Simpson.s inequality:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on $[a,b]$ and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in[a,b]}\left|f^{(4)}(x)\right|<\infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

For recent results on Simpson.s type inequalities see the papers [11-19].

Convexity on the co-ordinates can be given as following (see [10]);

Let us consider the bidimensional interval $\Delta=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. A function $f:\Delta\to\mathbb{R}$ will be called convex on the co-ordinates if the partial mappings $f_y:[a,b]\to\mathbb{R}$ $f_y(u)=f(u,y)$ and $f_x:[c,d]\to\mathbb{R}(u)$, $f_x(v)=f(x,v)$ are convex where defined for all $y\in[c,d]$ and $x\in[a,b]$.

Recall that the mapping $f:\Delta\to\mathbb{R}$ is convex on Δ, if the following inequality;

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le\lambda f(x,y)+(1-\lambda)f(z,w)$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

In [10], Dragomir proved the following inequalities:

Theorem 2. *Suppose that $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ is convex on the co-ordinates on Δ. Then one has the inequalities;*

$$\begin{aligned}&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\&\le\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\\&\le\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}.\end{aligned}\tag{1.1}$$

The above inequalities are sharp.

Recently, several papers have been written on the convex functions on the co-ordinates. Similar results can be found in [1-9] and [20,21,22,23].

In this paper, we will give Simpson-type inequalities for convex functions on the co-ordinates and bounded functions on the basis of the following lemma.

2. Main Results

To prove our main result, we need the following lemma.

Lemma 1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]$. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, then the following equality holds:*

$$\frac{\left[\begin{array}{l}f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{array}\right]}{9}$$

$$+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}$$
$$-\frac{1}{6(b-a)}\int_a^b\left[f(x,c)+4f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]dx$$
$$-\frac{1}{6(d-c)}\int_c^d\left[f(a,y)+4f\left(\frac{a+b}{2},y\right)+f(b,y)\right]dy \quad (2.1)$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$
$$=(b-a)(d-c)\int_0^1\int_0^1 p(t)q(s)\frac{\partial^2 f}{\partial t\partial s}\times(ta+(1-t)b,sc+(1-s)d)\,dtds$$

where

$$p(t)=\begin{cases}\left(t-\frac{1}{6}\right), & t\in\left[0,\frac{1}{2}\right]\\ \left(t-\frac{5}{6}\right), & t\in\left(\frac{1}{2},1\right]\end{cases}$$

and

$$q(s)=\begin{cases}\left(s-\frac{1}{6}\right), & t\in\left[0,\frac{1}{2}\right]\\ \left(s-\frac{5}{6}\right), & t\in\left(\frac{1}{2},1\right]\end{cases}.$$

Proof. Integrating by parts, we can write

$$\int_0^1\int_0^1 p(t)q(s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dtds$$
$$=\int_0^1 q(s)\left[\begin{array}{l}\int_0^{\frac{1}{2}}\left[\left(t-\frac{1}{6}\right)\frac{\partial^2 f}{\partial t\partial s}\times(ta+(1-t)b,sc+(1-s)d)\right]dt\\ +\int_{\frac{1}{2}}^1\left[\left(t-\frac{5}{6}\right)\frac{\partial^2 f}{\partial t\partial s}\times(ta+(1-t)b,sc+(1-s)d)\right]dt\end{array}\right]ds.$$

By integrating the right hand side of equality, we get

$$\int_0^1 q(s)\left\{\left[\left(t-\frac{1}{6}\right)\left(\frac{1}{a-b}\right)\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)\right]_0^{\frac{1}{2}}\right.$$
$$-\frac{1}{a-b}\int_0^{\frac{1}{2}}\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)dt$$
$$+\left[\left(t-\frac{5}{6}\right)\left(\frac{1}{a-b}\right)\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)\right]_{\frac{1}{2}}^1$$
$$\left.-\frac{1}{a-b}\int_{\frac{1}{2}}^1\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)dt\right\}ds$$
$$=\frac{1}{b-a}\left\{-\frac{1}{3}\int_0^{\frac{1}{2}}\left(s-\frac{1}{6}\right)\frac{\partial f}{\partial s}\left(\frac{a+b}{2},sc+(1-s)d\right)ds\right.$$
$$-\frac{1}{3}\int_{\frac{1}{2}}^1\left(s-\frac{5}{6}\right)\frac{\partial f}{\partial s}\left(\frac{a+b}{2},sc+(1-s)d\right)ds$$
$$-\frac{1}{6}\int_0^{\frac{1}{2}}\left(s-\frac{1}{6}\right)\frac{\partial f}{\partial s}(b,sc+(1-s)d)ds$$
$$-\frac{1}{6}\int_{\frac{1}{2}}^1\left(s-\frac{5}{6}\right)\frac{\partial f}{\partial s}(b,sc+(1-s)d)ds$$
$$+\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left(s-\frac{1}{6}\right)\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt$$
$$+\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left(s-\frac{5}{6}\right)\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt$$
$$-\frac{1}{6}\int_{\frac{1}{2}}^1\left(s-\frac{5}{6}\right)\frac{\partial f}{\partial s}(a,sc+(1-s)d)ds$$
$$-\frac{1}{3}\int_0^{\frac{1}{2}}\left(s-\frac{1}{6}\right)\frac{\partial f}{\partial s}\left(\frac{a+b}{2},sc+(1-s)d\right)ds$$
$$-\frac{1}{3}\int_{\frac{1}{2}}^1\left(s-\frac{5}{6}\right)\frac{\partial f}{\partial s}\left(\frac{a+b}{2},sc+(1-s)d\right)ds$$
$$-\frac{1}{6}\int_0^{\frac{1}{2}}\left(s-\frac{1}{6}\right)\frac{\partial f}{\partial s}(a,sc+(1-s)d)ds$$
$$+\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left(s-\frac{1}{6}\right)\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt$$
$$\left.+\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left(s-\frac{5}{6}\right)\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)dsdt\right\}.$$

Computing these integrals and using the change of the variable $x=ta+(1-t)b$ and $y=sc+(1-s)d$ for $(t,s)\in[0,1]^2$, then multiplying both sides with $(b-a)(d-c)$, we get the desired result.

Theorem 3. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping* $\Delta=[a,b]\times[c,d]$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is a convex function on the co-ordinates on* Δ *and* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$ *,then the following inequality holds*:

$$\left|\frac{\left[\begin{array}{l}f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{array}\right]}{9}\right.$$
$$+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}$$
$$\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\right|$$
$$\le\frac{25(b-a)(d-c)}{72}$$
$$\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{72}\right)$$

where

$$A=\frac{1}{6(b-a)}\int_a^b\left[f(x,c)+4f\left(x,\frac{c+d}{2}\right)+f(x,d)\right]dx+\frac{1}{6(d-c)}\int_c^d\left[f(a,y)+4f\left(\frac{a+b}{2},y\right)+f(b,y)\right]dy.$$

Proof. By using Lemma 1, we can write

$$\begin{aligned}&\left|\frac{\left[\begin{array}{l}f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{array}\right]}{9}\right.\\&+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\right|\\&\le(b-a)(d-c)\\&\times\int_0^1\int_0^1\left[\begin{array}{l}|p(t)q(s)|\\\times\left|\frac{\partial^2 f}{\partial t\partial s}\left((ta+(1-t)b),sc+(1-s)d\right)\right|\end{array}\right]dtds.\end{aligned}$$

Since $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ, we get

$$\begin{aligned}&\left|\frac{\left[\begin{array}{l}f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{array}\right]}{9}\right.\\&+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\right|\\&\le(b-a)(d-c)\\&\times\int_0^1|q(s)|\left[\int_0^1|p(t)|\left\{\begin{array}{l}t\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sc+(1-s)d\right)\right|\\+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sc+(1-s)d\right)\right|dt\end{array}\right\}\right]ds.\end{aligned}$$

Computing the integral in the right hand side of above inequality, we have

$$\begin{aligned}&\int_0^1|p(t)|\left\{\begin{array}{l}t\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sc+(1-s)d\right)\right|\\+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sc+(1-s)d\right)\right|\end{array}\right\}dt\\&=\int_0^{\frac16}\left(\frac16-t\right)\left\{\begin{array}{l}t\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sc+(1-s)d\right)\right|\\+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sc+(1-s)d\right)\right|\end{array}\right\}dt\\&+\int_{\frac16}^{\frac12}\left(t-\frac16\right)\left\{\begin{array}{l}t\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sc+(1-s)d\right)\right|\\+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sc+(1-s)d\right)\right|\end{array}\right\}dt\\&+\int_{\frac12}^{\frac56}\left(\frac56-t\right)\left\{\begin{array}{l}t\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sc+(1-s)d\right)\right|\\+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sc+(1-s)d\right)\right|\end{array}\right\}dt\\&+\int_{\frac56}^{1}\left(t-\frac56\right)\left\{\begin{array}{l}t\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sc+(1-s)d\right)\right|\\+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sc+(1-s)d\right)\right|\end{array}\right\}dt\\&=\frac{5}{72}\left(\begin{array}{l}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sc+(1-s)d\right)\right|\\+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sc+(1-s)d\right)\right|\end{array}\right).\end{aligned}$$

We obtain

$$\begin{aligned}&\left|\frac{\left[\begin{array}{l}f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{array}\right]}{9}\right.\\&+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A\right| \qquad (2.2)\\&\le\frac{5(b-a)(d-c)}{72}\\&\times\int_0^1|q(s)|\left\{\begin{array}{l}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sc+(1-s)d\right)\right|\\+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sc+(1-s)d\right)\right|\end{array}\right\}ds.\end{aligned}$$

By a similar argument for the above integral, we have

$$\begin{aligned}&\int_0^1|q(s)|\left\{\begin{array}{l}\left|\frac{\partial^2 f}{\partial t\partial s}\left(a,sc+(1-s)d\right)\right|\\+\left|\frac{\partial^2 f}{\partial t\partial s}\left(b,sc+(1-s)d\right)\right|\end{array}\right\}ds\\&=\int_0^{\frac16}\left(\frac16-s\right)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right\}ds\\&+\int_0^{\frac16}\left(\frac16-s\right)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}ds\end{aligned}$$

$$+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(s-\frac{1}{6}\right)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right\}ds$$
$$+\int_{\frac{1}{6}}^{\frac{1}{2}}\left(s-\frac{1}{6}\right)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}ds$$
$$+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-s\right)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right\}ds$$
$$+\int_{\frac{1}{2}}^{\frac{5}{6}}\left(\frac{5}{6}-s\right)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}ds$$
$$+\int_{\frac{5}{6}}^{1}\left(s-\frac{5}{6}\right)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right\}ds$$
$$+\int_{\frac{5}{6}}^{1}\left(s-\frac{5}{6}\right)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}ds$$
$$=\frac{5\left[\begin{array}{l}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\\+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\end{array}\right]}{72} \tag{2.3}$$

If we use (2.3) in (2.2), we get the required result.

Theorem 4. *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partial differentiable mapping on* $\Delta=[a,b]\times[c,d]$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is bounded, i.e.,*

$$\left\|\frac{\partial^2 f}{\partial t\partial s}\big((ta+(1-t)b),sc+(1-s)d\big)\right\|_\infty$$
$$=\sup_{(t,s)\in[0,1]^2}\left|\frac{\partial^2 f}{\partial t\partial s}\big((ta+(1-t)b),sc+(1-s)d\big)\right|<\infty$$

for all $(t,s)\in[0,1]^2$ *and* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$. *Then the following inequality holds*:

$$|\frac{\left[\begin{array}{l}f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{array}\right]}{9}$$
$$+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A|$$
$$\le\frac{25(b-a)(d-c)}{1296}$$
$$\times\left\|\frac{\partial^2 f}{\partial t\partial s}\big((ta+(1-t)b),sc+(1-s)d\big)\right\|_\infty$$

where A is as in Theorem 3.

Proof. From Lemma 1 and using the property of modulus, we have

$$|\frac{\left[\begin{array}{l}f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{array}\right]}{9}$$
$$+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A|$$
$$\le(b-a)(d-c)$$
$$\times\int_0^1\int_0^1|p(t)q(s)|\left|\frac{\partial^2 f}{\partial t\partial s}\big((ta+(1-t)b),sc+(1-s)d\big)\right|dtds.$$

Since $\frac{\partial^2 f}{\partial t\partial s}$ is bounded, we have

$$|\frac{\left[\begin{array}{l}f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{array}\right]}{9}$$
$$+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A| \tag{2.4}$$
$$\le(b-a)(d-c)$$
$$\times\left\|\frac{\partial^2 f}{\partial t\partial s}\big((ta+(1-t)b),sc+(1-s)d\big)\right\|_\infty$$
$$\times\int_0^1\int_0^1|p(t)q(s)|dtds.$$

By a simple calculation,

$$\int_0^1\int_0^1|p(t)q(s)|dtds=\frac{25}{1296}. \tag{2.5}$$

If we use (2.5) in (2.4), we have

$$|\frac{\left[\begin{array}{l}f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)+4f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\+f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)\end{array}\right]}{9}$$
$$+\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{36}$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx-A|$$
$$\le\frac{25(b-a)(d-c)}{1296}\left\|\frac{\partial^2 f}{\partial t\partial s}\big((ta+(1-t)b),sc+(1-s)d\big)\right\|_\infty.$$

This completes the proof.

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