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A New Proof of an Inequality for the Logarithm of the Gamma Function and Its Sharpness

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Abstract In the paper, the author shows that the partial sums $\gamma+\ln x+\frac{1}{2x}-\sum_{k=1}^{m}\frac{B_{2k}}{2kx^{2k}};m=1,2,$ are alternatively larger and smaller than the generalized Euler's harmonic numbers $H(x)=\psi(x+1)+\gamma$ with sharp bounds, where γ is the Euler's constant, B_i 's are the Bernoulli numbers and ψ is the digamma function.

Keywords: *Euler constant, ψ-function, harmonic numbers, inequalities, asymptotic expansion, sharp bounds*

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1. Introduction

Euler, in his "*Institutiones calculi differentialis*" [10], introduced the concept of inexplicable functions. These functions were appeared originally as functions in the positive integers of one symbol or more. He presented the following examples of the inexplicable functions:

$$F(x)=1\times2\times3\times\ldots\times x$$

$$H(x)=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x},$$

$$\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ldots+\frac{1}{x^n}$$

$$1+\frac{a-b}{a+2b}+\frac{a-2b}{a+3b}+\frac{a-3b}{a+4b}+\ldots+\frac{a-(x-1)b}{a+(x+1)b}$$

where there is no necessity for x to be an integer. The first function $F(x)$ generalizes the factorial function and the second one $H(x)$ generalizes the harmonic numbers

$$H_n=\sum_{k=1}^{n}\frac{1}{k};n\in N, \tag{1}$$

which are partial sums of the harmonic series. The function $H(x)$ can be defined by the definite integral

$$H(x)=\int_0^1\frac{1-y^x}{1-y}dy. \tag{2}$$

We will call the function $H(x)$, Euler's generalized Harmonic numbers. The recurrence relation of $H(x)$ is given by

$$H(x)=H(x-1)+\frac{1}{x} \tag{3}$$

and its reflection relation is

$$H(1-x)=H(x)+\pi\cot(\pi x)+\frac{2x-1}{x(1-x)}.$$

The multiplication formula is given by

$$H(nx)=H(x)+H(x-1/n)+H(x-2/n)$$
$$+\ldots+H(x-(n-1)/n)+\ln n;n\in N.$$

The function $H(x)$ is related to the Euler's constant γ by the relation

$$\gamma=\int_0^1 H(x)dx$$

and its relation with the digamma function $\psi(x)$ (the logarithmic derivative of the gamma function) is

$$\psi(x+1)=-\gamma+H(x). \tag{4}$$

There are many other generalizations of the harmonic numbers all of them depend only on the positive integers see [3,7,8,11,12,13,17,30,31]. Also, there are many estimations of the harmonic numbers H_n see [4,5,6,9,18,22,25,26,27,28,32,33].

A function f is said to be completely monotonic on an interval J if

$$(-1)^n f^{(n)}(x) \geq 0 \forall x \in J \text{ and } n = 0,1,2,\ldots. \tag{5}$$

If the inequality (5) is strict $\forall x \in J$ and all $n = 0,1,2,\ldots,$ then f is said to be strictly completely monotonic on the interval J.

In [1], Alzer proved that the functions

$$F_m(x) = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x$$
$$+x - \ln\sqrt{2\pi} - \sum_{j=1}^{2m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}$$

and

$$G_m(x) = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x$$
$$-x + \ln\sqrt{2\pi} - \sum_{j=1}^{2m+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}$$

For $m = 0,1,2,\ldots,$ are completely monotonic on $(0,\infty)$. This means that the functions $-F_m{}'(x)$ and $-G_m{}'(x)$ are also are completely monotonic on $(0,\infty)$ see [15]. These complete monotonicity have been repeated in [16] and [23]. These results can also be found in the survey [29]. From the complete monotonicity of $-F_m{}'(x)$ and $-G_m{}'(x)$, we get the following double inequality

$$\begin{aligned} -\sum_{k=1}^{2m+1} \frac{B_{2k}}{2kx^{2k}} &< \psi(x+1) - \ln x - \frac{1}{2x} \\ &< -\sum_{k=1}^{2m} \frac{B_{2k}}{2kx^{2k}}, \forall x \in (0,\infty); m = 0,1,2,\ldots. \end{aligned} \tag{6}$$

In this paper, we will present a new proof of the double inequality (6), using Artin's technique [2] and we will use a method due to Mortici [24] to prove that the bounds in (6) are the best possible. Also, we will provide new proof of the complete monotonicity of the two Functions $F_m(x)$ and $G_m(x)$.

2. Main Results

Theorem 1.
For x>0, $m = 0,1,2,\ldots,$

$$\sum_{k=1}^{2m} \frac{B_{2k}}{2kx^{2k}} < \gamma - H(x) + \ln x + \frac{1}{2x} < \sum_{k=1}^{2m+1} \frac{B_{2k}}{2kx^{2k}} \tag{7}$$

with sharp bounds.
Proof
Consider the function

$$r(x) = \gamma - H(x) + \ln x + \frac{1}{2x}; x > 0. \tag{8}$$

Then using the recurrence relation (3), we get

$$r(x) - r(x+1) = \frac{2x+1}{2x(x+1)} - \ln\left(\frac{x+1}{x}\right).$$

If we define the following function

$$g(x) = r(x) - r(x+1),$$

then

$$r(x) = \sum_{n=0}^{\infty} g(x+n).$$

We can express the function $g(x)$ by the integral representation

$$g(x) = \int_0^1 \frac{1/2 - v}{(x+v)^2} dv$$

and hence

$$r(x) = \sum_{n=0}^{\infty} \int_0^1 \frac{1/2 - v}{(x+n+v)^2} dv.$$

By considering the known discontinuous function

$$K(v) = \begin{cases} \frac{1}{2} - v & 0 < v < 1 \\ 0 & v = 0 \\ \text{periodic of period} 1 & \text{otherwise} \end{cases} \tag{9}$$

we get

$$r(x) = \sum_{n=0}^{\infty} \int_n^{n+1} \frac{K(v)}{(x+v)^2} dv = \lim_{n\to\infty} \int_0^n \frac{K(v)}{(x+v)^2} dv.$$

The oscillating of the function $\frac{K(v)}{(x+v)^2}$ and $\lim_{v\to\infty} \frac{K(v)}{(x+v)^2} = 0$ provided us by the existence of the integration

$$r(x) = \int_0^{\infty} \frac{K(v)}{(x+v)^2} dv. \tag{10}$$

In [2], Artin introduced the functions

$$K_{2n}(v) = 2(-1)^{n-1} \sum_{k=1}^{\infty} \frac{\cos(2k\pi v)}{(2k\pi)^{2n}} \tag{11}$$

$$K_{2n+1}(v) = 2(-1)^n \sum_{k=1}^{\infty} \frac{\sin(2k\pi v)}{(2k\pi)^{2n-1}}, \tag{12}$$

Where $-K_1(v)$ is the Fourier series [20,21] of the function $K(v)$. He showed that the series $K_n(v)$ is absolutely and uniformly convergent for all v with n=2,3,… and the series is uniformly convergent in every closed interval has no integer for n=1. Also,

$$\frac{d}{dv} K_{n+1}(v) = K_n(v) \tag{13}$$

for all v when n=2,3,… and for nonintegral v when n=1.

Now by repeated the integrations by parts of (10), we obtain

$$r(x)=\frac{K_2(0)1!}{x^2}+\frac{K_3(0)2!}{x^3}+\ldots+\frac{K_n(0)(n-1)!}{x^n}-\int_0^\infty\frac{K_n(v)n!}{(x+v)^{n+1}}dv. \tag{14}$$

But

$$\frac{K_n(0)(n-1)!}{x^n}-\int_0^\infty\frac{K_n(v)n!}{(x+v)^{n+1}}dv=\int_0^\infty\frac{(K_n(0)-K_n(v))n!}{(x+v)^{n+1}}dv. \tag{15}$$

If n is even, the $K_n(0)-K_n(v)$ has the same sign as $K_n(0)$ for all v and thus the integral (15) has this sign. Also,

$$K_{2n+1}(0)=0 \text{ and } K_{2n}(0)=\frac{B_{2n}}{(2n)!},\forall n. \tag{16}$$

Then the signs of $K_{2n}(0)$ alternate between minus and plus. Then the function $r(x)$ lies between any two successive partial sums

$$\frac{K_2(0)1!}{x^2}+\frac{K_4(0)3!}{x^4}+\ldots+\frac{K_{2n}(0)(2n-1)!}{x^{2n}} \tag{17}$$

In other words, for every n, there exists a number θ_n satisfies

$$r(x)=\frac{B_2}{2x^2}+\frac{B_4}{4x^4}+\ldots+\frac{B_{2n-2}}{(2n-2)x^{2n-2}}+\frac{B_{2n}}{2nx^{2n}}\theta_n,0<\theta_n<1 \tag{18}$$

and hence

$$\sum_{k=1}^{2m}\frac{B_{2k}}{2kx^{2k}}<\gamma-H(x)+\ln x+\frac{1}{2x}<\sum_{k=1}^{2m+1}\frac{B_{2k}}{2kx^{2k}}. \tag{19}$$

Now, we will prove the sharpness of the bounds in (19). By the definition of the asymptotic expansion [14], the expansion of a function $F(x)$ of the form

$$F(x)=g(x)+a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots$$

satisfies for every fixed k, that

$$\lim_{x\to\infty}x^k\left[F(x)-\left(g(x)+a_0+\frac{a_1}{x}+\ldots+\frac{a_k}{x^k}\right)\right]=0.$$

Then

$$\lim_{x\to\infty}x^{2m}\left[H(x)-\ln x-\gamma-\frac{1}{2x}+\sum_{k=1}^{2m-1}\frac{B_{2k}}{2kx^{2k}}\right]=-\frac{B_{2m}}{2m},m=1,2,3,\ldots. \tag{20}$$

If we have other constants $c_2,c_4,c_6,\ldots$ have the property that for all $n\in N$

$$-\frac{c_2}{x^2}-\frac{c_4}{x^4}-\frac{c_6}{x^6}<H(x)-\ln x-\gamma-\frac{1}{2x}<-\frac{c_2}{x^2}-\frac{c_4}{x^4},$$

$$-\frac{c_2}{x^2}-\frac{c_4}{x^4}-\frac{c_6}{x^6}-\frac{c_8}{x^8}-\frac{c_{10}}{x^{10}}<H(x)-\ln x-\gamma-\frac{1}{2x}<-\frac{c_2}{x^2}-\frac{c_4}{x^4}-\frac{c_6}{x^6}-\frac{c_8}{x^8},$$

$$-\frac{c_2}{x^2}-\frac{c_4}{x^4}-\frac{c_6}{x^6}-\frac{c_8}{x^8}-\frac{c_{10}}{x^{10}}-\frac{c_{12}}{x^{12}}-\frac{c_{14}}{x^{14}}<H(x)-\ln x-\gamma-\frac{1}{2x}<-\frac{c_2}{x^2}-\frac{c_4}{x^4}-\frac{c_6}{x^6}-\frac{c_8}{x^8}-\frac{c_{10}}{x^{10}}-\frac{c_{12}}{x^{12}},$$

etc. These inequalities give us that

$$\begin{aligned}&\lim_{x\to\infty}x^2\left[H(x)-\ln x-\gamma-\frac{1}{2x}\right]=-c_2,\\&\lim_{x\to\infty}x^4\left[H(x)-\ln x-\gamma-\frac{1}{2x}+\frac{c_2}{x^2}\right]=-c_4,\\&\lim_{x\to\infty}x^6\left[H(x)-\ln x-\gamma-\frac{1}{2x}+\frac{c_2}{x^2}+\frac{c_4}{x^4}\right]=-c_6,\end{aligned} \tag{21}$$

the relations (20) and (21), gives us that

$$c_{2l}=\frac{B_{2l}}{2l},\forall l\in N. \tag{22}$$

Then the choice of the constants $\frac{B_{2k}}{2k}$ in the inequality (7) is the best one. To complete our results, we need to prove that the constant 1/2 in the the function $H(x)-\ln x-\gamma-\frac{1}{2x}$ can not be improved by any method whatsoever. Consider the function

$$Z(x)=H(x)-\ln x-\gamma-\frac{A}{x},$$

then

$$Z(x)-Z(x+1)=\frac{-x-A}{x(x+1)}+\ln\left(\frac{x+1}{x}\right).$$

Now, let

$$V(x)=\frac{-x-A}{x(x+1)}+\ln\left(\frac{x+1}{x}\right);x>0$$

then

$$V'(x)=\frac{A(2x+1)-x}{x^2(x+1)^2}.$$

The The function $V(x)$ will be increasing if

$$A>\frac{x}{2x+1}=v(x)$$

and the function $v(x)$ is increasing function with $\lim_{x\to\infty}v(x)=1/2$. So, the best choice of A is 1/2. Also, the function $v(x)$ is increasing with limit tends to zero as x→∞, then

$$V(x)<0$$

Hence

$$Z(x)<Z(x+n), n\in N.$$

As n→∞, we get

$$Z(x)<0 \text{ or } H(x)-\ln x-\gamma<\frac{1}{2x}.$$

with sharp bound. Now, consider the function

$$W(x)=H(x)-\ln x-\gamma-\frac{1}{2x}+\frac{B}{x^2},$$

then

$$W(x+1)-W(x)=\frac{-2B(1+2x)+x+3x^2+2x^3}{2x^2(x+1)^2}+\ln\left(\frac{x}{x+1}\right).$$

Let

$$T(x)=\frac{-2B(1+2x)+x+3x^2+2x^3}{2x^2(x+1)^2}+\ln\left(\frac{x}{x+1}\right); x>0$$

then

$$T'(x)=\frac{4B\left(1+3x+3x^2\right)-x(1+x)}{3x^3(x+1)^3}.$$

The function $T(x)$ will be increasing if

$$B>\frac{x(1+x)}{4\left(1+3x+3x^2\right)}=t(x)$$

and the function $t(x)$ is increasing function with $\lim_{x\to\infty}t(x)=1/12$. So, the best choice of B is 1/12. Also, the function $T(x)$ is increasing with limit tends to zero as $x\to\infty$, then

$$T(x)<0$$

Hence

$$W(x+n)<W(x), n\in N.$$

As $n\to\infty$, we get

$$W(x)>0 \text{ or } H(x)-\ln x-\gamma>\frac{1}{2x}-\frac{1}{12x^2}.$$

Hence

$$\frac{1}{2x}-\frac{B_2}{x^2}<H(x)-\ln x-\gamma<\frac{1}{2x}$$

with sharp bounds.

As a special case we get the following result [19]

Corollary 2.1.

For any natural number $n\in N$,

$$\frac{1}{2n}-\sum_{k=1}^{2m+1}\frac{B_{2k}}{2kn^{2k}}<H_n-\ln n-\gamma<\frac{1}{2n}-\sum_{k=1}^{2m}\frac{B_{2k}}{2kn^{2k}}\quad m=0,1,2,3,\ldots \tag{23}$$

with sharp bounds.

Now, we will present a new proof for the complete monotonicity of the functions $F_m(x)$ and $G_m(x)$.

Lemma 2.2

For m=0,1,2,…, the functions

$$F_m(x)=\ln\Gamma(x)-\left(x-\frac{1}{2}\right)\ln x+x-\ln\sqrt{2\pi}-\sum_{j=1}^{2m}\frac{B_{2j}}{2j(2j-1)x^{2j-1}}$$

and

$$G_m(x)=-\ln\Gamma(x)+\left(x-\frac{1}{2}\right)\ln x-x+\ln\sqrt{2\pi}+\sum_{j=1}^{2m+1}\frac{B_{2j}}{2j(2j-1)x^{2j-1}}$$

are completely monotonic on (0,∞).

Proof

Using the relations (15), (17) and (16) at n=4m+2, we get

$$r(x)=\sum_{j=1}^{2m}\frac{B_{2j}}{2jx^{2j}}+\int_0^\infty\frac{\left(K_{4m+2}(0)-K_{4m+2}(v)\right)(4m+2)!}{(x+v)^{4m+3}}dv$$

and hence

$$(-1)^r\frac{d^r}{dx^r}\left(r(x)-\sum_{j=1}^{2m}\frac{B_{2j}}{2jx^{2j}}\right)=\int_0^\infty\frac{\left(K_{4m+2}(0)-K_{4m+2}(v)\right)(4m+2+r)!}{(x+v)^{4m+3+r}}dv,\quad r=1,2,3,\ldots. \tag{24}$$

But for m=0,1,2,3,…, the $K_{4m+2}(0)-K_{4m+2}(v)$ has the same sign as $K_{4m+2}(0)$ for all v and thus the integral (24) has this sign. Using the relation (11), we obtain

$$K_{4m+2}(0)=2(-1)^{2m}\sum_{k=1}^{\infty}\frac{1}{(2k\pi)^{2n}}>0$$

and hence

$$(-1)^r\frac{d^r}{dx^r}\left(r(x)-\sum_{j=1}^{2m}\frac{B_{2j}}{2jx^{2j}}\right)>0, r=1,2,3,\ldots.$$

But

$$\frac{d}{dx}F_m(x)=-\left(r(x)-\sum_{j=1}^{2m}\frac{B_{2j}}{2jx^{2j}}\right),$$

then

$$(-1)^r \frac{d^r}{dx^r} F_m(x) > 0, r = 0,1,2,\ldots.$$

Similarly, we can prove the complete monotonicity for $G(x)$ by replacing n by 4m+4.

Remark 1.

Series (17) is divergent so we can not take the limit as n tends to ∞ but the relation (18) provided us by approximations of the function $H(x)$ or any finite order. For example, if n=5 we obtain the following approximation:

$$H(x) = \gamma + \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4}$$
$$-\frac{1}{252x^6} + \frac{1}{240x^8} - \frac{\theta}{132x^{10}}, 0 < \theta < 1.$$

Remark 2.

The formula (10) implies the the function $r(x)$ is completely monotonic on $(0,\infty)$, that is

$$(-1)^r r^{(k)}(x) \geq 0, \forall k \in N; x \in (0,\infty).$$

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