Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 4, 113-118
Available online at http://pubs.sciepub.com/tjant/2/4/2

DOI:10.12691/tjant-2-4-2

On Some Inequalities for Functions Whose Second Derivatives Absolute Values Are S-Geometrically Convex

MEVLÜT TUNÇ[1,*], IBRAHİM KARABAYIR[2], EBRU YÜKSEL[3]

[1]Mustafa Kemal University, Faculty of Science and Arts, Department of Mathematics, Hatay, Turkey
[2]The Institute for Graduate Studies in Sciences and Engineering, Kilis 7 Aralk University, Kilis, Turkey
[3]AGRI Ibrahim Çeçen University, Faculty of Science and Arts, Department of Mathematics, AGRI, Turkey
*Corresponding author: mevluttttunc@gmail.com

Abstract In this paper, the authors achieve some new Hadamard type in- equalities using elementary well known inequalities for functions whose second derivatives absolute values are s-geometrically and geometrically convex. And also they get some applications for special means for positive numbers.

Keywords: *s-geometrically convex, geometrically convex, Hadamard.s inequality, Hölder.s inequality, power mean inequality, means*

Cite This Article: MEVLÜT TUNÇ, IBRAHİM KARABAYIR, and EBRU YÜKSEL, "On Some Inequalities for Functions Whose Second Derivatives Absolute Values Are S-Geometrically Convex." *Turkish Journal of Analysis and Number Theory*, vol. 2, no. 4 (2014): 113-118. doi: 10.12691/tjant-2-4-2.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with a < b. The following double inequalities:

$$f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}$$

hold. This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions.

In recent years many authors established several inequalities connected to this fact. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard-type inequalities see [5-9].

In this section we will present definitions and some results used in this paper.

Definition 1. Let I be an interval in $\mathbb{R}$: Then $f : I \to \mathbb{R}, \varnothing \ne I \subseteq \mathbb{R}$ is said to be convex if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y) \qquad (1.1)$$

for all $x,y \in I$ and $t \in [0,1]$.

Definition 2. [5] Let $s \in [0,1]$. A function $f : I \subset \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be s-convex in the second sense if

$$f(tx+(1-t)y) \le t^3 f(x)+(1-t)^3 f(y) \qquad (1.2)$$

for all $x, y \in I$ and $t \in [0,1]$.

It can be easily checked for s = 1, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

Recently, in [12], the concept of geometrically and s-geometrically convex functions was introduced as follows.

Definition 3. [12] A function $f : I \subset \mathbb{R}_+ = (0,\infty) \to \mathbb{R}_+$ is said to be a geometrically convex function if

$$f\left(x^t y^{1-t}\right) \le |f(x)|^t\, |f(y)|^{1-t} \qquad (1.3)$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 4. [12] A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a s-geometrically convex function if

$$f\left(x^t y^{1-t}\right) \le |f(x)|^{t^s}\, |f(y)|^{(1-t)^s} \qquad (1.4)$$

for some $s \in (0,1]$, where $x, y \in I$ and $t \in [0,1]$.

If s = 1, the s-geometrically convex function becomes a geometrically convex function on $\mathbb{R}_+$.

Example 1. [12] Let $f(x) = x^s / s$; $x \in (0,1]$, $0 < s < 1$, $q \ge 1$, and then the function

$$|f'(x)|^q = x^{(s-1)q} \qquad (1.5)$$

is monotonically decreasing on (0,1]. For $t \in [0,1]$, we have

$$(s-1)q\left(t^s - t\right) \le 0,\ \ (s-1)q\left((1-t)^s - (1-t)\right) \le 0 \qquad (1.6)$$

Hence, $\left|f'(x)\right|^q$ is s-geometrically convex on (0,1] for $0<s<1$.

2. Hadamardfis Type Inequalities

In order to prove our main theorems, we need the following lemma [1,3].

Lemma 1. [1,3] Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable mapping on I°, $a,b \in I$ with $a<b$ and f'' is integrable on [a,b], then the following equality holds:

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\\&=\frac{(b-a)^2}{2}\int_0^1 t(1-t)\,f''(ta+(1-t)b)\,dt\end{aligned} \tag{2.1}$$

A simple proof of this equality can be also done integrating by parts twice in the right hand side. The details are left to the interested reader.

The next theorems gives a new result of the upper Hermite-Hadamard inequality for s-geometrically and geometrically convex functions.

Theorem 1. Let $f: I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be twice differentiable mapping on I°, $a,b \in I$ with $a<b$ and f'' is integrable on [a,b] and $\left|f''(a)\right|\le 1$. If $\left|f''\right|$ is s-geometrically convex and monotonically decreasing on [a,b]; and $s\in(0,1]$ then the following inequality holds:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\le\frac{(b-a)^2}{12}\left|f''(b)\right|^s\left[\frac{\alpha(s,s)+1}{\left[\ln(\alpha(s,s))\right]^2}+\frac{2-2\alpha(s,s)}{\left[\ln(\alpha(s,s))\right]^3}\right]\end{aligned} \tag{2.2}$$

where

$$\alpha(u,\upsilon)=\left|f''(a)\right|^u\left|f''(b)\right|^{-\upsilon},\ u,\upsilon\ge 0 \tag{2.3}$$

Proof. Since $\left|f''\right|$ is a s-geometrically convex and monotonically decreasing on [a,b], from Lemma 1, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\le\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt\\&\le\int_0^1 t(1-t)\left|f''\left(a^t b^{(1-t)}\right)\right|dt\\&\le\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left(\left|f''(a)\right|^{t^s}\left|f''(b)\right|^{(1-t)^s}\right)dt\end{aligned}$$

If $0<\mu\le 1,\ 0<\alpha,\ s\le 1,$

$$\mu^{\alpha^s}\le\mu^{\alpha s} \tag{2.4}$$

When $\left|f''(a)\right|\le 1$, by(2.4), we get

$$\begin{aligned}&\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left(\left|f''(a)\right|^{t^s}\left|f''(b)\right|^{(1-t)^s}\right)dt\\&\le\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left(\left|f''(a)\right|^{st}\left|f''(b)\right|^{s(1-t)}\right)dt\\&\le\frac{(b-a)^2}{12}\left|f''(b)\right|^s\left[\frac{\left(\frac{f''(a)}{f''(b)}\right)^s+1}{\left(\ln\left|\frac{f''(a)}{f''(b)}\right|^s\right)^2}+\frac{2\left(1-\left(\frac{f''(a)}{f''(b)}\right)^s\right)}{\left(\ln\left|\frac{f''(a)}{f''(b)}\right|^s\right)^3}\right]\\&=\frac{(b-a)^2}{12}\left|f''(b)\right|^s\left[\frac{\alpha(s,s)+1}{\left[\ln(\alpha(s,s))\right]^2}+\frac{2-2\alpha(s,s)}{\left[\ln(\alpha(s,s))\right]^3}\right]\end{aligned}$$

which completes the proof.

Theorem 2. Let f $f: I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable on I°, $a,b \in I$ with $a<b$ and $f''\in L([a,b])$ and $\left|f''(a)\right|\le 1$. If $\left|f''\right|^q$ is s-geometrically convex and monotonically decreasing on [a,b] for p,q > 1 and $s\in(0,1]$; then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\le\frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left|f''(b)\right|^s\left(\Psi(\alpha(sq,sq))\right)^{\frac{1}{q}}\end{aligned} \tag{2.5}$$

where

$$\Psi(\alpha)=\begin{cases}1 & \alpha=1\\ \dfrac{\alpha-1}{\ln\alpha} & \alpha\ne 1\end{cases}$$

$$\alpha(u,\upsilon)=\left|f''(a)\right|^{\mu}\left|f''(b)\right|^{-\upsilon},\ u,\upsilon\ge 0$$

Proof. Since $\left|f''\right|^q$ is a s-geometrically convex and monotonically decreasing on [a,b], from Lemma 1 and the well known Hölder inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\&\le\frac{(b-a)^2}{2}\int_0^1\left|t(1-t)\right|\left|f''(ta+(1-t)b)\right|dt\\&\le\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''\left(a^t b^{(1-t)}\right)\right|dt\\&\le\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left(\left|f''(a)\right|^{t^s}\left|f''(b)\right|^{(1-t)^s}\right)dt\\&\le\frac{(b-a)^2}{2}\left(\int_0^1\left[t(1-t)\right]^p dt\right)\left(\int_0^1\left|f''(a)\right|^{t^s}\left|f''(b)\right|^{(1-t)^s}dt\right)^{\frac{1}{q}}\end{aligned} \tag{2.6}$$

If $\left|f''(a)\right|\le 1$, by (2.4), we obtain

$$\begin{aligned}&\int_0^1 \left|f''(a)\right|^{qt^s}\left|f''(b)\right|^{q(1-t)^s}dt\\ &\le \int_0^1 \left|f''(a)\right|^{sqt}\left|f''(b)\right|^{sq(1-t)}dt\\ &=\left|f''(a)\right|^{sq}\int_0^1\left(\left|f''(a)\right|\left|f''(b)\right|^{-1}\right)^{sqt}dt\\ &=\left|f''(b)\right|^{sq}\Psi\left(\alpha\left(sq,sq\right)\right),\end{aligned} \tag{2.7}$$

and then from (2.6)-(2.7), (2.8) holds.

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le \frac{(b-a)^2}{2}\left(\int_0^1[t(1-t)]^p\right)^{\frac{1}{p}}\left(\left|\int_0^1 f''(a)\right|^{t^s}\left|f''(b)\right|^{(1-t)^s}dt\right)^{\frac{1}{q}}\\ &\le \frac{(b-a)^2}{2}\left(\frac{2^{-1-2p}\sqrt{\pi}\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left(\left|f''(b)\right|^{sq}\Psi\left(\alpha\left(sq,sq\right)\right)\right)^{\frac{1}{q}}\\ &=\frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left(\left|f''(b)\right|^{sq}\Psi\left(\alpha\left(sq,sq\right)\right)\right)^{\frac{1}{q}}\end{aligned} \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$. We have to note that, the Beta and Gamma Functions (see [1]), are described respectively, as follows.

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x,y>0$$

and

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt,\ x>0$$

are used to evaluate the integral

$$\int_0^1\left[t(1-t)\right]^p dt=\int_0^1 t^p(1-t)^p dt=\beta(p+1,p+1)$$

Using the proprieties of Beta function, that is, $\beta(x,x)=2^{1-2x}\beta\left(\frac{1}{2},x\right)$ and $\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we can achieve that

$$\begin{aligned}\beta(p+1,p+1)&=2^{1-2(p+1)}\beta\left(\frac{1}{2},p+1\right)\\ &=2^{-1-2p}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\end{aligned}$$

where $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, which completes the proof.

Corollary 1. Let $f: I\subseteq(0,\infty)\to(0,\infty)$ be differentiable on I°, $a,b\in I$ with $a<b$ and $f''\in L([a,b])$. If $\left|f''\right|^q$ is s-geometrically convex and monotonically decreasing on [a,b] for $p,q>1$ and $s\in(0,1]$, then

i) When $p = q = 2$, one has

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le\frac{(b-a)^2}{2\sqrt{30}}\left|f''(b)\right|^s\left(\Psi(2s,2s)\right)^{\frac{1}{2}}\end{aligned}$$

where $\Gamma(3)=2,\Gamma\left(\frac{7}{2}\right)=\frac{15\sqrt{\pi}}{8}$.

ii) If we take $s = 1$ in (2.5), we have for geometrically convex, one has

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le\frac{(b-a)^2}{8}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left|f''(b)\right|\left(\Psi(q,q)\right)^{\frac{1}{q}}\end{aligned}$$

Theorem 3. Let $f: I\subset\mathbb{R}_+\to\mathbb{R}_+$ be twice differentiable on I°, $a,b\in I$ with $a<b$ and $f''\in L([a,b])$ and $\left|f''(a)\right|\le 1$. If $\left|f''(x)\right|^q$ is s-geometrically convex and monotonically decreasing on [a,b], for $q\ge 1$ and $s\in(0,1]$, then

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le\frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left|f''(b)\right|^s\\ &\left[\frac{\alpha(sq,sq)+1}{\left[\ln\left(\alpha(sq,sq)\right)\right]^2}+\frac{2-2\alpha(sq,sq)}{\left[\ln\left(\alpha(sq,sq)\right)\right]^3}\right]\end{aligned} \tag{2.9}$$

where α (u,v) is same with above (2.3).

Proof. Since $\left|f''\right|^q$ is a s-geometrically convex and monotonically decreasing on [a,b], from Lemma 1 and the well known power mean integral inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\le\frac{(b-a)^2}{2}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left|f''\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\\ &\le\frac{(b-a)^2}{2}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}\\ &\left(\int_0^1 t(1-t)\left(\left|f''(a)\right|^{t^s}\left|f''(b)\right|^{(1-t)^s}\right)^q dt\right)^{\frac{1}{q}}\\ &\le\frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left|f''(a)\right|^{qt^s}\left|f''(b)\right|^{q(1-t)^s}dt\right)^{\frac{1}{q}}\end{aligned}$$

When $\left|f''(a)\right|\le 1$, by (2.4), we get

$$\begin{aligned}
&\int_0^1 t(1-t)\left|f''(a)\right|^{qt^s}\left|f''(b)\right|^{q(1-t)^s}dt\\
&\le \int_0^1 t(1-t)\left|f''(a)\right|^{sqt}\left|f''(b)\right|^{sq(1-t)}dt\\
&\le \left|f''(a)\right|^{sq}\int_0^1 t(1-t)\left|f''(a)\right|^{sqt}\left|f''(b)\right|^{-sqt}dt\\
&=\left|f''(b)\right|^{sq}\int_0^1 t(1-t)\left|\frac{f''(a)}{f''(b)}\right|^{sqt}dt\\
&=\left|f''(b)\right|^{sq}\left[\frac{\left|\frac{f''(a)}{f''(b)}\right|^{sq}+1}{\left(\ln\left|\frac{f''(a)}{f''(b)}\right|^{sq}\right)^2}+\frac{2\left(1-\left|\frac{f''(a)}{f''(b)}\right|^{sq}\right)}{\left(\ln\left|\frac{f''(a)}{f''(b)}\right|^{sq}\right)^3}\right]
\end{aligned}$$

which completes the proof.

Theorem 4. Let $f: I\subseteq(0,\infty)\to(0,\infty)$ be twice differentiable on I°, $a,b\in I$ with $a<b$ and $f''\in L([a,b])$ and $\left|f''(a)\right|\le 1$. If $\left|f''\right|$ is s-geometrically convex and monotonically decreasing on [a,b] for $\mu,\eta>0$ with $\mu+\eta=1$ and $s\in(0,1]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{4}\left(\mu\frac{\sqrt{\pi}\,\Gamma\left(1+\frac{1}{\mu}\right)}{4^{\frac{1}{\mu}}\Gamma\left(\frac{3}{2}+\frac{1}{\mu}\right)}+\eta\left|f''(b)\right|^{\frac{s}{\eta}}\Psi\left(\alpha\left(\frac{s}{\eta},\frac{s}{\eta}\right)\right)\right)$$

where $\alpha(u,v)$ is same with above (2.3) and

$$\Psi(\alpha)=\begin{cases}1 & \alpha=1\\ \dfrac{\eta(\alpha-1)}{s\ln\alpha} & \alpha\ne 1\end{cases}$$

Proof. Since $\left|f''\right|$ is a s-geometrically convex and monotonically decreasing on [a,b], from Lemma 1, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(ta+(1-t)b)\right|dt \qquad (2.10)\\
&\le\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''\left(a^t b^{1-t}\right)\right|dt\\
&\le\frac{(b-a)^2}{2}\int_0^1 t(1-t)\left|f''(a)\right|^{t^s}\left|f''(b)\right|^{(1-t)^s}dt
\end{aligned}$$

for all $t\in[0,1]$. Using the well known inequality $mn\le\mu m^{\frac{1}{\mu}}+\eta n^{\frac{1}{\eta}}$, on the right side of (2.10), we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\frac{(b-a)^2}{2}\left[\begin{array}{l}\mu\int_0^1(t(1-t))^{\frac{1}{\mu}}dt\\ +\eta\int_0^1\left(\left|f''(a)\right|^{t^s}\left|f''(b)\right|^{(1-t)^s}\right)^{\frac{1}{\eta}}dt\end{array}\right]
\end{aligned}$$

When $\left|f''(a)\right|\le 1$, by (2.4), we get

$$\begin{aligned}
&\int_0^1\left|f''(a)\right|^{\frac{t^s}{\eta}}\left|f''(b)\right|^{\frac{(1-t)^s}{\eta}}dt\\
&\le\int_0^1\left|f''(a)\right|^{\frac{st}{\eta}}\left|f''(b)\right|^{\frac{s(1-t)}{\eta}}dt\\
&\le\left|f''(b)\right|^{\frac{s}{\eta}}\int_0^1\left(\left|f''(a)\right|\left|f''(b)\right|^{-1}\right)^{\frac{st}{\eta}}dt\\
&=\left|f''(b)\right|^{\frac{s}{\eta}}\Psi\left(\alpha\left(\frac{s}{\eta},\frac{s}{\eta}\right)\right),
\end{aligned}$$

and then, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\le\frac{(b-a)^2}{2}\left[\begin{array}{l}\mu\int_0^1(t(1-t))^{\frac{1}{\mu}}dt\\ +\eta\int_0^1\left(\left|f''(a)\right|^{t^s}\left|f''(b)\right|^{(1-t)^s}\right)^{\frac{1}{\eta}}dt\end{array}\right]\\
&\le\frac{(b-a)^2}{2}\left(\begin{array}{l}\mu\dfrac{2^{-1-\frac{2}{\mu}}\sqrt{\pi}\,\Gamma\left(1+\frac{1}{\mu}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{\mu}\right)}\\ +\mu\left|f''(b)\right|^{\frac{s}{\eta}}\Psi\left(\alpha\left(\frac{s}{\eta},\frac{s}{\eta}\right)\right)\end{array}\right)\\
&=\frac{(b-a)^2}{4}\left(\begin{array}{l}\mu\dfrac{\sqrt{\pi}\,\Gamma\left(1+\frac{1}{\mu}\right)}{4^{\frac{1}{\mu}}\Gamma\left(\frac{3}{2}+\frac{1}{\mu}\right)}\\ +\eta\left|f''(b)\right|^{\frac{s}{\eta}}\Psi\left(\alpha\left(\frac{s}{\eta},\frac{s}{\eta}\right)\right)\end{array}\right)
\end{aligned}$$

We have to note that, using the Beta and Gamma Functions and evaluating the integral, we get

$$\int_0^1(t(1-t))^{\frac{1}{\mu}}dt=\int_0^1 t^{\frac{1}{\mu}}(1-t)^{\frac{1}{\mu}}dt=\beta\left(\frac{1}{\mu}+1,\frac{1}{\mu}+1\right)$$

And, using the proprieties of Beta function, that is, $\beta(x,x)=2^{1-2x}\beta\left(\frac{1}{2},x\right)$ and $\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we achieve

$$\beta\left(\frac{1}{\mu}+1,\frac{1}{\mu}+1\right)=2^{1-2\left(\frac{1}{\mu}+1\right)}\beta\left(\frac{1}{2},\frac{1}{\mu}+1\right)$$

$$=2^{-1-\frac{2}{\mu}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(1+\frac{1}{\mu}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{\mu}\right)}$$

Where $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, which completes the proof.

3. Applications to Special Means for Positive Numbers

Let

$$A(a,b)=\frac{a+b}{2},L(a,b)=\frac{b-a}{\ln b-\ln a}(a\neq b),$$

$$L_p(a,b)=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p},a\neq b,p\in\mathbb{R},p\neq -1,0$$

be the arithmetic, logarithmic, generalized logarithmic means for a, b > 0 respectively.

Proposition 1. Let $0<a<b\leq 1,\ 0<s\leq 1$. Then, we have

$$\left|A\left(a^{s+1},b^{s+1}\right)-L_{s+1}^{s+1}(a,b)\right|$$

$$\leq\frac{(b-a)^2}{12}s(s+1)b^{(s-1)s}\left[\frac{\left|\frac{a^{(s-1)}}{b^{(s-1)}}\right|^s+1}{\left[\ln\left|\frac{a^{(s-1)}}{b^{(s-1)}}\right|^s\right]^2}+\frac{2\left(1-\left|\frac{a^{(s-1)}}{b^{(s-1)}}\right|^s\right)}{\left[\ln\left|\frac{a^{(s-1)}}{b^{(s-1)}}\right|^s\right]^3}\right]$$

Proof. The assertion follows from Theorem 1 applied to s-geometrically convex mapping $f(x)=\frac{x^{s+1}}{s(s+1)}$, $x\in(0,1]$.

Proposition 2. Let $0<a<b\leq 1, 0<s\leq 1$, and $q\geq 1$. Then, we have

$$\left|A\left(a^{s+1},b^{s+1}\right)-L_{s+1}^{s+1}(a,b)\right|$$

$$\leq\frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}s(s+1)b^{(s-1)s}$$

$$\left[\frac{\left|\frac{a^{(s-1)}}{b^{(s-1)}}\right|^{sq}+1}{\left[\ln\left|\frac{a^{(s-1)}}{b^{(s-1)}}\right|^{qs}\right]^2}+\frac{2\left(1-\left|\frac{a^{(s-1)}}{b^{(s-1)}}\right|^{sq}\right)}{\left[\ln\left|\frac{a^{(s-1)}}{b^{(s-1)}}\right|^{sq}\right]^3}\right]$$

Proof. The assertion follows from Theorem 2 applied to s-geometrically convex mapping $f(x)=\frac{x^{s+1}}{s(s+1)}$, $x\in(0,1]$.

Proposition 3. Let $0<a<b\leq 1, 0<s\leq 1$, and $q\geq 1$. Then, we have

$$\left|A\left(a^{s+1},b^{s+1}\right)-L_{s+1}^{s+1}(a,b)\right|$$

$$\leq\frac{(b-a)^2}{8}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)$$

$$s(s+1)\left(b^{(s-1)s}\right)^s\left(\Psi\left(\alpha(sq,sq)\right)\right)^{\frac{1}{q}}$$

In here, $\alpha=\frac{|f''(a)|}{|f''(b)|}=\left|\frac{a^{s-1}}{b^{s-1}}\right|=\left(\frac{a}{b}\right)^{s-1}$ and we can write this; if $a=b,\alpha=1$ and $\Psi(\alpha)=1$, and if $a\neq b,\alpha\neq 1$ and $\Psi(\alpha)=\frac{\left(\frac{a}{b}\right)^{\frac{(s-1)s}{n}}}{\frac{(s-1)s}{n}\ln\left(\frac{a}{b}\right)}$.

Proof. The assertion follows from Theorem 3 applied to s-geometrically convex mapping $f(x)=\frac{x^{s+1}}{s(s+1)}$, $x\in(0,1]$.

Proposition 4. Let Let $0<a<b\leq 1, 0<s\leq 1$; and $q\geq 1$. Then, we have

$$\left|A\left(a^{s+1},b^{s+1}\right)-L_{s+1}^{s+1}(a,b)\right|$$

$$\leq\frac{s(s+1)(b-a)^2}{4}\left(\mu\frac{\sqrt{\pi}\Gamma\left(1+\frac{1}{\mu}\right)}{4^{\frac{1}{\mu}}\Gamma\left(\frac{3}{2}+\frac{1}{\mu}\right)}+\eta\left(b^{s-1}\right)^{\frac{s}{\eta}}\Psi\left(\alpha\left(\frac{s}{\eta},\frac{s}{\eta}\right)\right)\right)$$

In here, $\alpha=\frac{|f''(a)|}{|f''(b)|}=\left|\frac{a^{s-1}}{b^{s-1}}\right|=\left(\frac{a}{b}\right)^{s-1}$ and we can write this; if a = b, $\alpha=1$ and d $\Psi(\alpha)=1$, and if $a\neq b,\alpha\neq 1$ and $\Psi(\alpha)=\frac{\eta\left(\left(\frac{a}{b}\right)^{s-1}-1\right)}{(s-1)s\ln\left(\frac{a}{b}\right)}$.

Proof. The assertion follows from Theorem 4 applied to s-geometrically convex mapping $f(x)=\frac{x^{s+1}}{s(s+1)}$, $x\in(0,1]$.

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