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Determinantal Identities of Fibonacci, Fibonacci Like and Lucas Numbers

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Abstract Determinants have played a significant part in various areas in mathematics. For instance, they are quite useful in the analysis and solution of system of linear equations. There are different perspectives on the study of determinant. In this paper we present some determinant identities of Fibonacci and Lucas numbers.

Keywords: *fibonacci number, lucas number, fibonacci like sequence*

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1. Introduction

It is well known that the Fibonacci numbers and polynomials are of great importance in the study of many subjects such as algebra, geometry, combinatorics, approximation theory, graph theory and number theory itself.

They occur in a variety of other fields such as finance, art, architecture, music, etc.

One may notice several practical and effective instruments for calculating determinants in the nice survey articles [6] and [9]. Much attention has been paid to the evaluation of determinants of matrices, especially when their entries are given recursively [7]. There is a long tradition of using matrices and determinants to study Fibonacci numbers. Bicknell – Johnson and Spears [2] use elementary matrix operationand determinants to generate classes of identities for generalized Fibonacci numbers. Cahill and Narayan [4] show how Fibonacci and Lucas numbers arise as determinants of some tridiagonal matrices. T. Benjamin, T. Cameron and J. Quinn [1], provides combinatorial interpretations for Fibonacci identities using determinants. Koshy [5] explained two chapters on the useof matrices and determinants in Fibonacci numbers.

Spivey [9] describe the sum property for determinants and presented new proofs of identities like the Cassini identity, the d'Ocagne identity and the Catalan identity.

Macfarlane [8] use the property for determinants to give new identities involving Fibonacci and related number is defined by the recurrence relation,

$$F_n = F_{n-1} + F_{n-2}, n \geq 2 \textit{ with } \mathrm{F}_0 = 0, F_1 = 1$$

Lucas sequence is defined by the recurrence relation,

$$L_n = L_{n-1} + L_{n-2}, n \geq 2 \textit{ with } \mathrm{L}_0 = 2, \mathrm{L}_1 = 1$$

The Fibonacci- Like sequence [3] is defined by recurrence relation,

$$M_{n+2} = M_{n+1} + M_n, n \geq 2 \textit{ with } \mathrm{M}_0 = 2, \mathrm{M}_1 = 2$$

first few numbers of Fibonacci-Like sequences are 2, 2, 4, 6, 10, 16, and so on

Theorem 1 For every integer $n \geq 0$:

$$\begin{vmatrix} 1 & M_{n+3} & M_{n+3}^2 \\ 1 & M_{n+2} & M_{n+2}^2 \\ 1 & M_{n+1} & M_{n+1}^2 \end{vmatrix} = \left(M_{n+1} - M_{n+2}\right)\left(M_{n+2} - M_{n+3}\right)\left(M_{n+3} - M_{n+1}\right)$$

Theorem 2 For every integer $n \geq 0$:

$$\begin{vmatrix} 1 & F_{n+3} & F_{n+3}^2 \\ 1 & F_{n+2} & F_{n+2}^2 \\ 1 & F_{n+1} & F_{n+1}^2 \end{vmatrix} = \left(F_{n+1} - F_{n+2}\right)\left(F_{n+2} - F_{n+3}\right)\left(F_{n+3} - F_{n+1}\right)$$

Theorem 3 For every integer $n \geq 0$:

$$\begin{vmatrix} 1 & M_{n+1} & M_{n+1}^3 \\ 1 & M_{n+2} & M_{n+2}^3 \\ 1 & M_{n+3} & M_{n+3}^3 \end{vmatrix} = \left(M_{n+1} - M_{n+2}\right)\left(M_{n+2} - M_{n+3}\right) \times \left(M_{n+3} - M_{n+1}\right)\left(M_{n+1} - M_{n+2} + M_{n+3}\right)$$

Theorem 4 For every integer $n \geq 0$**:**

$$\begin{vmatrix} 1 & F_{n+1} & F_{n+1}^3 \\ 1 & F_{n+2} & F_{n+2}^3 \\ 1 & F_{n+3} & F_{n+3}^3 \end{vmatrix}$$
$$= (F_{n+1} - F_{n+2})(F_{n+2} - F_{n+3})$$
$$\times (F_{n+3} - F_{n+1})(F_{n+1} - F_{n+2} + F_{n+3})$$

Theorem 5 For every integer n ≥ 0:

$$\begin{vmatrix} 1 & L_{n+3} & L_{n+3}^2 \\ 1 & L_{n+2} & L_{n+2}^2 \\ 1 & L_{n+1} & L_{n+1}^2 \end{vmatrix}$$
$$= -(L_{n+1} - L_{n+2})(L_{n+2} - L_{n+3})(L_{n+3} - L_{n+1})$$

Theorem 6 For every integer n ≥ 0:

$$\begin{vmatrix} 1 & L_{n+1} & L_{n+1}^3 \\ 1 & L_{n+2} & L_{n+2}^3 \\ 1 & L_{n+3} & L_{n+3}^3 \end{vmatrix}$$
$$= (L_{n+1} - L_{n+2})(L_{n+2} - L_{n+3})$$
$$\times (L_{n+3} - L_{n+1})(L_{n+1} - L_{n+2} + L_{n+3})$$

Theorem 7 For every integer n ≥ 0:

$$\begin{vmatrix} F_n & F_{n+1} - F_{n+2} & F_{n+2} + F_{n+1} \\ F_n + F_{n+2} & F_{n+2} & F_{n+2} - F_n \\ F_n - F_{n+1} & F_n + F_{n+1} & F_{n+2} \end{vmatrix}$$
$$= (F_n + F_{n+1} + F_{n+2})\left(F_n^2 + F_{n+1}^2 + F_{n+2}^2\right)$$

Theorem 8 For every integer n ≥ 0:

$$\begin{vmatrix} L_n & L_{n+1} - L_{n+2} & L_{n+2} + L_{n+1} \\ L_n + L_{n+2} & L_{n+2} & L_{n+2} - L_n \\ L_n - L_{n+1} & L_n + L_{n+1} & L_{n+2} \end{vmatrix}$$
$$= (L_n + L_{n+1} + L_{n+2})\left(L_n^2 + L_{n+1}^2 + L_{n+2}^2\right)$$

Theorem 9 For every integer n ≥ 0:

$$\begin{vmatrix} M_n & M_{n+1} - M_{n+2} & M_{n+2} + M_{n+1} \\ M_n + M_{n+2} & M_{n+2} & M_{n+2} - M_n \\ M_n - M_{n+1} & M_n + M_{n+1} & M_{n+2} \end{vmatrix}$$
$$= (M_n + M_{n+1} + M_{n+2})\left(M_n^2 + M_{n+1}^2 + M_{n+2}^2\right)$$

Theorem 10 For every integer n ≥ 0:

$$\begin{vmatrix} 1 & F_n & F_n^2 \\ 1 & F_{n+1} & F_{n+1}^2 \\ 1 & F_{n+2} & F_{n+2}^2 \end{vmatrix} = (F_n - F_{n+1})(F_{n+1} - F_{n+2})(F_{n+2} - F_n)$$

Theorem 11 For every integer n ≥ 0:

$$\begin{vmatrix} 1 & M_n & M_n^2 \\ 1 & M_{n+1} & M_{n+1}^2 \\ 1 & M_{n+2} & M_{n+2}^2 \end{vmatrix} = \begin{pmatrix} M_n \\ -M_{n+1} \end{pmatrix}\begin{pmatrix} M_{n+1} \\ -M_{n+2} \end{pmatrix}\begin{pmatrix} M_{n+2} \\ -M_n \end{pmatrix}$$

Theorem 12 For every integer n ≥ 0**:**

$$\begin{vmatrix} 1 & L_n & L_n^2 \\ 1 & L_{n+1} & L_{n+1}^2 \\ 1 & L_{n+2} & L_{n+2}^2 \end{vmatrix} = (L_n - L_{n+1})(L_{n+1} - L_{n+2})(L_{n+2} - L_n)$$

Theorem 13 For every integer n ≥ 0:

$$\begin{vmatrix} F_n & F_{n+1} & F_n + F_{n+1} \\ F_{n+1} & F_{n+2} & F_{n+1} + F_{n+2} \\ F_n + F_{n+1} & F_{n+1} + F_{n+2} & 0 \end{vmatrix} = 0$$

Theorem 14 For every integer n ≥ 0:

$$\begin{vmatrix} L_n & L_{n+1} & L_n + L_{n+1} \\ L_{n+1} & L_{n+2} & L_{n+1} + L_{n+2} \\ L_n + L_{n+1} & L_{n+1} + L_{n+2} & 0 \end{vmatrix} = 0$$

Theorem 15 For every integer n ≥ 0:

$$\begin{vmatrix} 1 + F_n & 1 & 1 \\ 1 & 1 + F_{n+1} & 1 \\ 1 & 1 & 1 + F_{n+2} \end{vmatrix}$$
$$= F_n.F_{n+1} + F_{n+1}.F_{n+2}$$
$$+ F_{n+2}.F_n + F_n F_{n+1} F_{n+2}$$

Theorem 16 For every integer n ≥ 0:

$$\begin{vmatrix} 1 + L_n & 1 & 1 \\ 1 & 1 + L_{n+1} & 1 \\ 1 & 1 & 1 + L_{n+2} \end{vmatrix}$$
$$= L_n.\mathrm{L}_{n+1} + L_{n+1}.\mathrm{L}_{n+2}$$
$$+ L_{n+2}.\mathrm{L}_n + L_n L_{n+1} L_{n+2}$$

Theorem 17 For every integer n ≥ 0:

$$\begin{vmatrix} 1 + M_n & 1 & 1 \\ 1 & 1 + M_{n+1} & 1 \\ 1 & 1 & 1 + M_{n+2} \end{vmatrix}$$
$$= M_n.\mathrm{M}_{n+1} + M_{n+1}.\mathrm{M}_{n+2}$$
$$+ M_{n+2}.\mathrm{M}_n + M_n M_{n+1} L_{n+2}$$

Theorem 18 For every integer n ≥ 0:

$$\begin{vmatrix} 1 & F_n^2 + F_{n+1}.F_{n+2} & F_n^2 \\ 1 & F_{n+1}^2 + F_{n+2}.F_n & F_{n+1}^2 \\ 1 & F_{n+2}^2 + F_n.F_{n+1} & F_{n+2}^2 \end{vmatrix}$$
$$= (F_n + F_{n+1} + F_{n+2})\left(F_n^2 + F_{n+1}^2 + F_{n+2}^2\right)$$

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