

A Note on Saigo's Fractional Integral Inequalities

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Abstract In this paper, some new integral inequalities related to the bounded functions, involving Saigo's fractional integral operators, are established. Special cases of the main results are also pointed out.

Keywords: Integral inequalities, Gauss hypergeometric function, Saigo's fractional integral operators

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1. Introduction and Preliminaries

Under various assumptions (Chebyshev inequality, Grüss inequality, Minkowski inequality, Hermite-Hadamard inequality, Ostrowski inequality etc.), inequalities are playing a very significant role in all fields of mathematics, particularly in the theory of approximations (see [2,6,7,13,14,17,23]). Therefore, in the literature we found several extensions and generalizations of these integral inequalities for the functions of bounded variation, synchronous, Lipschitzian, monotonic, absolutely continuous and n-times differentiable mappings etc. ([10,11,12,15,16,19,20,21,22,26,27,28]). In the past recent years, one more dimension have been added to this study, by introducing number of integral inequalities involving various fractional calculus and q-calculus operators. For detailed account, one may refer [1,3,4,5,8,9,18,24,25,29-35] and the references cited therein.

Recently, Tariboon et al. [33] investigated certain new integral inequalities for the integrable functions, whose bounds are also integrable functions, involving the Riemann-Liouville fractional integral operators. Our aim in this paper, is to obtain a general extensions of the results due to Tariboon et al. [33]. Main results investigated here provide certain new integral inequalities associated with the integrable functions, whose bounds are also integrable functions, involving the Saigo's fractional integral operators. We also give some consequent results and special cases of the main results.

Firstly, we mention below the basic definitions and notations of some well-known operators of fractional calculus, which shall be used in the sequel.

Let $\alpha > 0, \beta, \eta \in \mathbb{R}$, then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}$ of order α for a real-valued continuous function $f(t)$ is defined by ([36], see also [37], p. 19):

$$I_{0,t}^{\alpha,\beta,\eta} f(t) = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau) d\tau, \quad (1.1)$$

where, the function ${}_2F_1(-)$ appearing as a kernel for the operator (1.1) is the Gaussian hypergeometric function defined by

$${}_2F_1(a, b; c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (1.2)$$

and $(a)_n$ is the Pochhammer symbol

$$(a)_n = a(a+1)\dots(a+n-1), \quad (a)_0 = 1.$$

The operator $I_{0,t}^{\alpha,\beta,\eta}$ includes both the Riemann-Liouville and the Erdélyi-Kober fractional integral operators given by the following relationships:

$$I^\alpha f(t) = I_{0,t}^{\alpha,-\alpha,0} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad (\alpha > 0) \quad (1.3)$$

and

$$I^{\alpha,\eta} f(t) = I_{0,t}^{\alpha,0,\eta} f(t) = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^\eta f(\tau) d\tau \quad (\alpha > 0, \eta \in \mathbb{R}). \quad (1.4)$$

Following [36], for $f(t) = t^\mu$ in (1.1), we get

$$I_{0,t}^{\alpha,\beta,\eta} t^\mu = \frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)} t^{\mu-\beta} \quad (\alpha > 0, \min(\mu, \mu-\beta+\eta) > -1, t > 0). \quad (1.5)$$

2. Main Results

In this section, we obtain certain integral inequalities, related to the integrable functions, whose bounds are also integrable functions, involving Saigo's fractional hypergeometric operators. The results are given in the form of the following theorems:

Theorem 1. Let f, φ_1 , and φ_2 are integrable functions defined on $[0, \infty)$, such that

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t), \text{ for all } t \in [0, \infty). \quad (2.1)$$

Then, for $t > 0$, we have

$$\begin{aligned} & I_{0,t}^{\gamma, \delta, \zeta} \varphi_1(t) I_{0,t}^{\alpha, \beta, \eta} f(t) + I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) I_{0,t}^{\gamma, \delta, \zeta} f(t) \\ & \geq I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) I_{0,t}^{\gamma, \delta, \zeta} \varphi_1(t) + I_{0,t}^{\alpha, \beta, \eta} f(t) I_{0,t}^{\gamma, \delta, \zeta} f(t), \end{aligned} \quad (2.2)$$

where $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$, $\gamma > \max\{0, -\delta\}$, $\delta < 1$ and $\delta - 1 < \zeta < 0$.

Proof. By the hypothesis of inequality (2.1), for any $\tau, \rho > 0$, we have

$$(\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \geq 0,$$

which follows that

$$\begin{aligned} & \varphi_2(\tau) f(\rho) + \varphi_1(\rho) f(\tau) \\ & \geq \varphi_1(\rho) \varphi_2(\tau) + f(\tau) f(\rho). \end{aligned} \quad (2.3)$$

Consider

$$\begin{aligned} F(t, \tau) &= \frac{t^{-\alpha-\beta} (t-\tau)^{\alpha-1}}{\Gamma(\alpha)} {}_2F_1\left(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{t}\right) \\ & \quad (\tau \in (0, t); t > 0) \\ &= \frac{1}{\Gamma(\alpha)} \frac{(t-\tau)^{\alpha-1}}{t^{\alpha+\beta}} + \frac{(\alpha+\beta)(-\eta)(t-\tau)^\alpha}{\Gamma(\alpha+1) t^{\alpha+\beta+1}} \\ & \quad + \frac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)(t-\tau)^{\alpha+1}}{2\Gamma(\alpha+2) t^{\alpha+\beta+2}} + \dots, \end{aligned} \quad (2.4)$$

which remains positive, for all $\tau \in (0, t) (t > 0)$, under the conditions stated with Theorem 1. Multiplying both sides of (2.3) by $F(t, \tau)$ (where $F(t, \tau)$ is given by (2.4)) and integrating the resulting identity with respect to τ from 0 to t , and using (1.1), we get

$$\begin{aligned} & f(\rho) I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) + \varphi_1(\rho) I_{0,t}^{\alpha, \beta, \eta} f(t) \\ & \geq \varphi_1(\rho) I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) + f(\rho) I_{0,t}^{\alpha, \beta, \eta} f(t). \end{aligned} \quad (2.5)$$

Next, on multiplying both sides of (2.5) by

$$\begin{aligned} H(t, \rho) &= \frac{t^{-\gamma-\delta} (t-\rho)^{\gamma-1}}{\Gamma(\gamma)} {}_2F_1(\gamma + \delta, -\zeta; \gamma; 1 - \rho) \\ & \quad (\rho \in (0, t); t > 0), \end{aligned} \quad (2.6)$$

which also remains positive, for all $\rho \in (0, t) (t > 0)$. Upon integrating the resulting inequality so obtained with respect to ρ from 0 to t , and using the operator (1.1), we easily arrive at the desired result (2.1).

It may be noted that, for $\gamma = \alpha$, $\delta = \beta$, $\zeta = \eta$, the Theorem 1 immediately reduces to the following result:

Corollary 1. Let φ_1 and φ_2 are integrable functions defined on $[0, \infty)$ and satisfying inequality (2.1). Then, for $t > 0$, we have

$$\begin{aligned} & I_{0,t}^{\alpha, \beta, \eta} \varphi_1(t) I_{0,t}^{\alpha, \beta, \eta} f(t) + I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) I_{0,t}^{\alpha, \beta, \eta} f(t) \\ & \geq I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) I_{0,t}^{\alpha, \beta, \eta} \varphi_1(t) + \left(I_{0,t}^{\alpha, \beta, \eta} f(t) \right), \end{aligned} \quad (2.7)$$

where $\alpha > \max\{0, -\beta\}$, $\beta < 1$ and $\beta - 1 < \eta < 0$.

Theorem 2. Let f and g be two integrable functions on $[0, \infty)$ and $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are four integrable functions on $[0, \infty)$, such that

$$\begin{aligned} & \varphi_1(t) \leq f(t) \leq \varphi_2(t), \quad \psi_1(t) \leq g(t) \leq \psi_2(t), \\ & \text{for all } t \in [0, \infty). \end{aligned} \quad (2.8)$$

Then, for $t > 0$, $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$, $\gamma > \max\{0, -\delta\}$, $\delta < 1$ and $\delta - 1 < \zeta < 0$, the following inequalities holds true:

$$\begin{aligned} & I_{0,t}^{\gamma, \delta, \zeta} \psi_1(t) I_{0,t}^{\alpha, \beta, \eta} f(t) + I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) I_{0,t}^{\gamma, \delta, \zeta} g(t) \\ & \geq I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) I_{0,t}^{\gamma, \delta, \zeta} \psi_1(t) + I_{0,t}^{\alpha, \beta, \eta} f(t) I_{0,t}^{\gamma, \delta, \zeta} g(t), \end{aligned} \quad (2.9)$$

$$\begin{aligned} & I_{0,t}^{\gamma, \delta, \zeta} \varphi_1(t) I_{0,t}^{\alpha, \beta, \eta} g(t) + I_{0,t}^{\alpha, \beta, \eta} \psi_2(t) I_{0,t}^{\gamma, \delta, \zeta} f(t) \\ & \geq I_{0,t}^{\alpha, \beta, \eta} \psi_2(t) I_{0,t}^{\gamma, \delta, \zeta} \varphi_1(t) + I_{0,t}^{\alpha, \beta, \eta} g(t) I_{0,t}^{\gamma, \delta, \zeta} f(t), \end{aligned} \quad (2.10)$$

$$\begin{aligned} & I_{0,t}^{\gamma, \delta, \zeta} \varphi_2(t) I_{0,t}^{\alpha, \beta, \eta} \psi_2(t) + I_{0,t}^{\alpha, \beta, \eta} f(t) I_{0,t}^{\gamma, \delta, \zeta} g(t) \\ & \geq I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) I_{0,t}^{\gamma, \delta, \zeta} g(t) + I_{0,t}^{\alpha, \beta, \eta} f(t) I_{0,t}^{\gamma, \delta, \zeta} \psi_2(t), \end{aligned} \quad (2.11)$$

$$\begin{aligned} & I_{0,t}^{\gamma, \delta, \zeta} \varphi_1(t) I_{0,t}^{\alpha, \beta, \eta} \psi_1(t) + I_{0,t}^{\alpha, \beta, \eta} f(t) I_{0,t}^{\gamma, \delta, \zeta} g(t) \\ & \geq I_{0,t}^{\alpha, \beta, \eta} \varphi_1(t) I_{0,t}^{\gamma, \delta, \zeta} f(t) + I_{0,t}^{\alpha, \beta, \eta} f(t) I_{0,t}^{\gamma, \delta, \zeta} \psi_1(t). \end{aligned} \quad (2.12)$$

Proof. Let f and g are two integrable functions and satisfying inequality (2.8), then to prove (2.9), we can write

$$(\varphi_2(\tau) - f(\tau))(g(\rho) - \psi_1(\rho)) \geq 0,$$

which follows that

$$\begin{aligned} & \varphi_2(\tau) g(\rho) + \psi_1(\rho) f(\tau) \\ & \geq \psi_1(\rho) \varphi_2(\tau) + f(\tau) g(\rho). \end{aligned} \quad (2.13)$$

On multiplying both sides of (2.13) by $F(t, \tau)$ (where $F(t, \tau)$ is given by (2.4)) and integrating with respect to τ from 0 to t , then by making use of (1.1), we get

$$\begin{aligned} & g(\rho) I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) + \psi_1(\rho) I_{0,t}^{\alpha, \beta, \eta} f(t) \\ & \geq \psi_1(\rho) I_{0,t}^{\alpha, \beta, \eta} \varphi_2(t) + g(\rho) I_{0,t}^{\alpha, \beta, \eta} f(t). \end{aligned} \quad (2.14)$$

Next, multiplying both sides of (2.13) by $H(t, \rho) (\rho \in (0, t), t > 0)$, where $H(t, \rho)$ is given by (2.6), and integrating with respect to ρ from 0 to t , we easily arrive at the desired result (2.9).

Following the similar procedure, one can easily establish the remaining inequalities (2.10) to (2.12) by using the following inequalities, respectively

$$\begin{aligned}(\psi_2(\tau) - g(\tau))(f(\rho) - \varphi_1(\rho)) &\geq 0, \\ (\varphi_2(\tau) - f(\tau))(g(\rho) - \psi_2(\rho)) &\geq 0\end{aligned}$$

and

$$(\varphi_1(\tau) - f(\tau))(g(\rho) - \psi_1(\rho)) \geq 0.$$

Therefore, we omit the further details of the proof of these results.

3. Consequent Results and Special Cases

The Saigo's fractional integral operator defined by (1.1), possess the advantage that the Erdélyi-Kober and the Riemann-Liouville type fractional integral operators happen to be the particular cases of this operator. Therefore, by suitably specializing the parameters, we now briefly consider some special cases of the result derived in the preceding section. To this end, let us set $\beta = 0$ and $\delta = 0$, and make use of the relation (1.4), then Theorems 1 & 2 yields the following inequalities involving the Erdélyi-Kober type fractional integral operators:

Corollary 2. Let f, φ and φ_2 are integrable functions defined on $[0, \infty)$ and satisfying inequality (2.1), then for $t > 0$, we have

$$\begin{aligned}I^{\gamma, \zeta} \varphi_1(t) I^{\alpha, \eta} f(t) + I^{\alpha, \eta} \varphi_2(t) I^{\gamma, \zeta} f(t) \\ \geq I^{\alpha, \eta} \varphi_2(t) I^{\gamma, \zeta} \varphi_1(t) + I^{\alpha, \eta} f(t) I^{\gamma, \zeta} f(t),\end{aligned}\quad (3.1)$$

where $\alpha > 0, -1 < \eta < 0, \gamma > 0$ and $-1 < \zeta < 0$.

Corollary 3. Let f and g be two integrable functions on $[0, \infty)$ and $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are four integrable functions on $[0, \infty)$, and satisfying inequality (2.8). Then, for $t > 0, \alpha > 0, -1 < \eta < 0, \gamma > 0$ and $-1 < \zeta < 0$, the following inequalities holds true:

$$\begin{aligned}I^{\gamma, \zeta} \psi_1(t) I^{\alpha, \eta} f(t) + I^{\alpha, \eta} \varphi_2(t) I^{\gamma, \zeta} g(t) \\ \geq I^{\alpha, \eta} \varphi_2(t) I^{\gamma, \zeta} \psi_1(t) + I^{\alpha, \eta} f(t) I^{\gamma, \zeta} g(t),\end{aligned}\quad (3.2)$$

$$\begin{aligned}I^{\gamma, \zeta} \varphi_1(t) I^{\alpha, \eta} g(t) + I^{\alpha, \eta} \psi_2(t) I^{\gamma, \zeta} f(t) \\ \geq I^{\alpha, \eta} \psi_2(t) I^{\gamma, \zeta} \varphi_1(t) + I^{\alpha, \eta} g(t) I^{\gamma, \zeta} f(t),\end{aligned}\quad (3.3)$$

$$\begin{aligned}I^{\gamma, \zeta} \varphi_2(t) I^{\alpha, \eta} \psi_2(t) + I^{\alpha, \eta} f(t) I^{\gamma, \zeta} g(t) \\ \geq I^{\alpha, \eta} \varphi_2(t) I^{\gamma, \zeta} g(t) + I^{\alpha, \eta} f(t) I^{\gamma, \zeta} \psi_2(t),\end{aligned}\quad (3.4)$$

$$\begin{aligned}I^{\gamma, \zeta} \varphi_1(t) I^{\alpha, \eta} \psi_1(t) + I^{\alpha, \eta} f(t) I^{\gamma, \zeta} g(t) \\ \geq I^{\alpha, \eta} \varphi_1(t) I^{\gamma, \zeta} f(t) + I^{\alpha, \eta} f(t) I^{\gamma, \zeta} \psi_1(t).\end{aligned}\quad (3.5)$$

Next, if we replace β by $-\alpha, \delta$ by $-\gamma$ and make use of the relation (1.3), then Theorems 1 & 2 corresponds to the known integral inequalities involving Riemann-Liouville type fractional integral operators, due to Tariboon *et al* [33].

Further, if we put $\varphi_1(t) = m, \varphi_2(t) = M, \psi_1(t) = p$ and $\psi_2(t) = P$, where $m, M, p, P \in \mathbb{R}, \forall t \in [0, \infty)$ and make use of formula (1.5), then the Theorems 1 & 2 leads to the following particular results:

Corollary 4. Let f be an integrable function defined on $[0, \infty)$, such that

$$m \leq f(t) \leq M, \quad m, M \in \mathbb{R} \quad \text{for all } t \in [0, \infty). \quad (3.6)$$

Then, for $t > 0$, we have

$$\begin{aligned}m \frac{\Gamma(1-\delta+\zeta)t^{-\delta}}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} I_{0,t}^{\alpha, \beta, \eta} f(t) \\ + M \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\gamma, \delta, \zeta} f(t) \\ \geq mM \frac{\Gamma(1-\beta+\eta)\Gamma(1-\delta+\zeta)t^{-\beta-\delta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} \\ + I_{0,t}^{\alpha, \beta, \eta} f(t) I_{0,t}^{\gamma, \delta, \zeta} f(t),\end{aligned}\quad (3.7)$$

where $\alpha > \max\{0, -\beta\}, \beta < 1, \beta - 1 < \eta < 0, \gamma > \max\{0, -\delta\}, \delta < 1$ and $\delta - 1 < \zeta < 0$.

Corollary 5. Let f and g be two integrable functions on $[0, \infty)$, such that

$$\begin{aligned}m \leq f(t) \leq M, \quad p \leq g(t) \leq P, \\ m, p, M, P \in \mathbb{R} \quad \text{for all } t \in [0, \infty).\end{aligned}\quad (3.8)$$

Then, for $t > 0, \alpha < \max\{0, -\beta\}, \beta < 1, \beta - 1 < \mu < 0, \gamma > \max\{0, -\delta\}, \delta < 1$ and $\delta - 1 < \zeta < 0$, the following inequalities holds true:

$$\begin{aligned}p \frac{\Gamma(1-\delta+\zeta)t^{-\delta}}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} I_{0,t}^{\alpha, \beta, \eta} f(t) \\ + M \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\gamma, \delta, \zeta} g(t) \\ \geq pM \frac{\Gamma(1-\beta+\eta)\Gamma(1-\delta+\zeta)t^{-\beta-\delta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} \\ + I_{0,t}^{\alpha, \beta, \eta} f(t) I_{0,t}^{\gamma, \delta, \zeta} g(t),\end{aligned}\quad (3.9)$$

$$\begin{aligned}m \frac{\Gamma(1-\delta+\zeta)t^{-\delta}}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} I_{0,t}^{\alpha, \beta, \eta} g(t) \\ + P \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\gamma, \delta, \zeta} f(t) \\ \geq mP \frac{\Gamma(1-\beta+\eta)\Gamma(1-\delta+\zeta)t^{-\beta-\delta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} \\ + I_{0,t}^{\alpha, \beta, \eta} g(t) I_{0,t}^{\gamma, \delta, \zeta} f(t),\end{aligned}\quad (3.10)$$

$$\begin{aligned}MP \frac{\Gamma(1-\beta+\eta)\Gamma(1-\delta+\zeta)t^{-\beta-\delta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} \\ + I_{0,t}^{\alpha, \beta, \eta} f(t) I_{0,t}^{\gamma, \delta, \zeta} g(t)\end{aligned}$$

$$\geq M \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\gamma,\delta,\zeta} g(t) \quad (3.11)$$

$$\begin{aligned} & + P \frac{\Gamma(1-\delta+\zeta)t^{-\delta}}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} I_{0,t}^{\alpha,\beta,\eta} f(t), \\ & mP \frac{\Gamma(1-\beta+\eta)\Gamma(1-\delta+\zeta)t^{-\beta-\delta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} \\ & + I_{0,t}^{\alpha,\beta,\eta} f(t) I_{0,t}^{\gamma,\delta,\zeta} g(t) \\ & \geq m \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} I_{0,t}^{\gamma,\delta,\zeta} g(t) \quad (3.12) \\ & + P \frac{\Gamma(1-\delta+\zeta)t^{-\delta}}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)} I_{0,t}^{\alpha,\beta,\eta} f(t). \end{aligned}$$

Again, if we set $\varphi_1(t) = t$ and $\varphi_2(t) = t+1$ and make use of formula (1.5), then the Theorem 1 and Corollary 1, further leads to the following integral inequalities:

Corollary 6. Let f be an integrable function defined on $[0, \infty)$, such that

$$t \leq f(t) \leq t+1, \text{ for all } t \in [0, \infty)$$

Then, for $t > 0, \alpha > \max\{0, -\beta\}, \beta < 1, \beta-1 < \eta < 0, \gamma > \max\{0, \delta\}, \delta < 1$ and $\delta-1 < \zeta < 0$ we have

$$\begin{aligned} & \frac{\Gamma(2-\delta+\zeta)t^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\gamma+\zeta)} I_{0,t}^{\alpha,\beta,\eta} f(t) \\ & + \left(\frac{\Gamma(2-\beta+\eta)t^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} + \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} \right) I_{0,t}^{\gamma,\delta,\zeta} f(t) \\ & \geq \left(\frac{\Gamma(2-\beta+\eta)t^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} + \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} \right) \left(\frac{\Gamma(2-\delta+\zeta)t^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\gamma+\zeta)} \right) \\ & + I_{0,t}^{\alpha,\beta,\eta} f(t) I_{0,t}^{\gamma,\delta,\zeta} f(t). \end{aligned} \quad (3.13)$$

Corollary 7. Let f be an integrable function defined on $[0, \infty)$, such that

$$t \leq f(t) \leq t+1, \text{ for all } t \in [0, \infty)$$

Then, for $t > 0, \alpha > \max\{0, -\beta\}, \beta < 1$ and $\beta-1 < \eta < 0$, we have

$$\begin{aligned} & \left(\frac{2\Gamma(2-\beta+\eta)t^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} + \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} \right) I_{0,t}^{\alpha,\beta,\eta} f(t) \\ & \geq \left(I_{0,t}^{\alpha,\beta,\eta} f(t) \right)^2 \quad (3.14) \\ & + \left(\frac{\Gamma(2-\beta+\eta)t^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} + \frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)} \right) \times \left(\frac{\Gamma(2-\beta+\eta)t^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} \right). \end{aligned}$$

In this paper, we have introduced certain general integral inequalities, related to the integrable and bounded functions f and g , involving Saigo's fractional integral operators. Therefore, we conclude with the remark that, by suitably specializing the arbitrary function $\varphi_1(t), \varphi_2(t), \psi_1(t)$ and $\psi_2(t)$, one can further easily obtain additional integral inequalities involving the Riemann-Liouville, Erd'elyi-Kober and Saigo type fractional integral operators from our main results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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