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Application of Parseval's Theorem on Evaluating Some Definite Integrals

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Abstract This paper uses the mathematical software Maple for the auxiliary tool to study six types of definite integrals. We can determine the infinite series forms of these definite integrals by using Parseval's theorem. On the other hand, we provide some definite integrals to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions using Maple.

Keywords: *definite integrals, infinite series forms, Parseval's theorem, Maple*

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1. Introduction

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Mozart, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research.

In calculus and engineering mathematics courses, we learnt many methods to solve the integral problems including change of variables method, integration by parts method, partial fractions method, trigonometric substitution method, and so on. In this paper, we study the following six types of definite integrals which are not easy to obtain their answers using the methods mentioned above.

$$\int_0^{2\pi} \sinh^2(r\cos x)\cdot\cos^2(r\sin x)dx \tag{1}$$

$$\int_0^{2\pi} \cosh^2(r\cos x)\cdot\sin^2(r\sin x)dx \tag{2}$$

$$\int_0^{2\pi} [\sinh^2(r\cos x)+\sin^2(r\sin x)]dx \tag{3}$$

$$\int_0^{2\pi} \cosh^2(r\cos x)\cdot\cos^2(r\sin x)dx \tag{4}$$

$$\int_0^{2\pi} \sinh^2(r\cos x)\cdot\sin^2(r\sin x)dx \tag{5}$$

$$\int_0^{2\pi} [\cosh^2(r\cos x)+\cos^2(r\sin x)]dx \tag{6}$$

where r is any real number. We can obtain the infinite series forms of these definite integrals by using Parseval's theorem; these are the major results of this paper (i.e., Theorems 1 and 2). As for the study of related integral problems can refer to [1-17]. On the other hand, we propose some definite integrals to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce a notation and a definition and some formulas used in this article.

2.1. Notation

Let $z=a+ib$ be a complex number, where $i=\sqrt{-1}$, a,b are real numbers. We denote a the real part of z by $\mathrm{Re}(z)$, and b the imaginary part of z by $\mathrm{Im}(z)$.

2.2. Definition

Suppose $f(x)$ is a continuous function defined on $[0,2\pi]$, then the Fourier series expansion of $f(x)$ is $\frac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$, where $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx$, and $a_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos kxdx$, $b_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin kxdx$ for all positive integers k.

2.3. Formulas

2.3.1. Euler's Formula

$e^{ix}=\cos x+i\sin x$, where x is any real number.

2.3.2. DeMoivre's Formula

$(\cos x+i\sin x)^n=\cos nx+i\sin nx$, where n is any integer, and x is any real number.

2.3.3. Taylor Series Expansion of Hyperbolic Sine Function ([18])

$\sinh(z)=\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}$, where z is any complex number.

2.3.4. Taylor Series Expansion of Hyperbolic Cosine Function ([18])

$\cosh(z)=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}$, where z is any complex number.

Next, we introduce an important theorem used in this study.

2.4. Parseval's Theorem ([19])

If $f(x)$ is a continuous function defined on $[0,2\pi]$, and $f(0)=f(2\pi)$. If the Fourier series expansion of $f(x)$ is $\frac{1}{2}a_0+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$, then $\frac{1}{\pi}\int_0^{2\pi}f^2(x)dx=\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)$.

Before deriving the first major result of this paper, we need a lemma.

2.5. Lemma 1

Suppose p,q are any real numbers. Then

$$\sinh(p+iq)=\sinh p\cdot\cos q+i\cosh p\cdot\sin q \tag{7}$$

$$\sinh^2 p\cdot\cos^2 q+\cosh^2 p\cdot\sin^2 q=\sinh^2 p+\sin^2 q \tag{8}$$

Proof $\sinh(p+iq)$

$$=\frac{1}{2}[e^{p+iq}-e^{-(p+iq)}]$$

$$=\frac{1}{2}[e^p(\cos q+i\sin q)-e^{-p}(\cos q-i\sin q)]$$

$$=\frac{1}{2}(e^p-e^{-p})\cos q+i\frac{1}{2}(e^p+e^{-p})\sin q$$

$$=\sinh p\cdot\cos q+i\cosh p\cdot\sin q$$

And

$$\sinh^2 p\cdot\cos^2 q+\cosh^2 p\cdot\sin^2 q$$

$$=\sinh^2 p\cdot(1-\sin^2 q)+\cosh^2 p\cdot\sin^2 q$$

$$=\sinh^2 p+\sin^2 q$$

Next, we determine the infinite series forms of the definite integrals (1), (2) and (3).

2.6. Theorem 1

Suppose r is any real number. Then the definite integrals

$$\int_0^{2\pi}\sinh^2(r\cos x)\cdot\cos^2(r\sin x)dx=\pi\cdot\sum_{k=0}^{\infty}\frac{r^{4k+2}}{[(2k+1)!]^2} \tag{9}$$

$$\int_0^{2\pi}\cosh^2(r\cos x)\cdot\sin^2(r\sin x)dx=\pi\cdot\sum_{k=0}^{\infty}\frac{r^{4k+2}}{[(2k+1)!]^2} \tag{10}$$

$$\int_0^{2\pi}[\sinh^2(r\cos x)+\sin^2(r\sin x)]dx=2\pi\cdot\sum_{k=0}^{\infty}\frac{r^{4k+2}}{[(2k+1)!]^2} \tag{11}$$

Proof Because

$$\sinh(r\cos x)\cdot\cos(r\sin x)$$

$$=\operatorname{Re}[\sinh(re^{ix})] \quad (By\ (7))$$

$$=\operatorname{Re}\left[\sum_{k=0}^{\infty}\frac{(re^{ix})^{2k+1}}{(2k+1)!}\right] \quad (By\ Formula\ 2.3.3.)$$

$$=\operatorname{Re}\left[\sum_{k=0}^{\infty}\frac{r^{2k+1}e^{i(2k+1)x}}{(2k+1)!}\right] \quad (By\ DeMoivre's\ formula)$$

$$=\sum_{k=0}^{\infty}\frac{r^{2k+1}}{(2k+1)!}\cos(2k+1)x \quad (By\ Euler's\ formula) \tag{12}$$

It follows that

$$\int_0^{2\pi}\sinh^2(r\cos x)\cdot\cos^2(r\sin x)dx=\pi\cdot\sum_{k=0}^{\infty}\frac{r^{4k+2}}{[(2k+1)!]^2}$$

(Using (12) and Parseval's theorem)
Similarly, because

$$\cosh(r\cos x)\cdot\sin(r\sin x)$$

$$=\operatorname{Im}[\sinh(re^{ix})] \quad (By\ (7))$$

$$=\operatorname{Im}\left[\sum_{k=0}^{\infty}\frac{(re^{ix})^{2k+1}}{(2k+1)!}\right]$$

$$= \sum_{k=0}^{\infty} \frac{r^{2k+1}}{(2k+1)!} \sin(2k+1)x \tag{13}$$

It follows that

$$\int_0^{2\pi} \cosh^2(r\cos x)\cdot\sin^2(r\sin x)dx$$
$$= \pi\cdot\sum_{k=0}^{\infty} \frac{r^{4k+2}}{[(2k+1)!]^2}$$

(Using (13) and Parseval's theorem)
On the other hand, from the summation of (9) and (10) and using (8), we obtain

$$\int_0^{2\pi} [\sinh^2(r\cos x)+\sin^2(r\sin x)]dx$$
$$= 2\pi\cdot\sum_{k=0}^{\infty} \frac{r^{4k+2}}{[(2k+1)!]^2}$$

Before deriving the second major result of this study, we also need a lemma.

2.7. Lemma 2

Suppose p, q are any real numbers. Then

$$\cosh(p+iq) = \cosh p\cdot\cos q + i\sinh p\cdot\sin q \tag{14}$$

$$\cosh^2 p\cdot\cos^2 q + \sinh^2 p\cdot\sin^2 q = \sinh^2 p + \cos^2 q \tag{15}$$

Proof $\cosh(p+iq)$

$$= \frac{1}{2}[e^{p+iq} + e^{-(p+iq)}]$$
$$= \frac{1}{2}[e^p(\cos q + i\sin q) + e^{-p}(\cos q - i\sin q)]$$
$$= \frac{1}{2}(e^p + e^{-p})\cos q + i\frac{1}{2}(e^p - e^{-p})\sin q$$
$$= \cosh p\cdot\cos q + i\sinh p\cdot\sin q$$

And

$$\cosh^2 p\cdot\cos^2 q + \sinh^2 p\cdot\sin^2 q$$
$$= (1+\sinh^2 p)\cdot\cos^2 q + \sinh^2 p\cdot\sin^2 q$$
$$= \sinh^2 p + \cos^2 q$$

Finally, we find the infinite series forms of the definite integrals (4), (5) and (6).

2.8. Theorem 2

Suppose r is any real number. Then the definite integrals

$$\int_0^{2\pi} \cosh^2(r\cos x)\cdot\cos^2(r\sin x)dx$$
$$= 2\pi + \pi\sum_{k=1}^{\infty} \frac{r^{4k}}{[(2k)!]^2} \tag{16}$$

$$\int_0^{2\pi} \sinh^2(r\cos x)\cdot\sin^2(r\sin x)dx$$
$$= \pi\cdot\sum_{k=1}^{\infty} \frac{r^{4k}}{[(2k)!]^2} \tag{17}$$

$$\int_0^{2\pi} [\sinh^2(r\cos x)+\cos^2(r\sin x)]dx$$
$$= 2\pi + 2\pi\cdot\sum_{k=1}^{\infty} \frac{r^{4k}}{[(2k)!]^2} \tag{18}$$

Proof Because

$$\cosh(r\cos x)\cdot\cos(r\sin x)$$
$$= \mathrm{Re}[\cosh(re^{ix})] \qquad (By\ (14))$$
$$= \mathrm{Re}\left[\sum_{k=0}^{\infty} \frac{(re^{ix})^{2k}}{(2k)!}\right] \qquad (By\ Formula\ 2.3.4.) \tag{19}$$
$$= 1 + \sum_{k=1}^{\infty} \frac{r^{2k}}{(2k)!}\cos 2kx$$
$$(By\ DeMoivre's\ formula\ and\ Euler's\ formula)$$

Using (19) and Parseval's theorem, we have

$$\int_0^{2\pi} \cosh^2(r\cos x)\cdot\cos^2(r\sin x)dx$$
$$= 2\pi + \pi\sum_{k=1}^{\infty} \frac{r^{4k}}{[(2k)!]^2}$$

Similarly, because

$$\sinh(r\cos x)\cdot\sin(r\sin x)$$
$$= \mathrm{Im}[\cosh(re^{ix})] \qquad (By\ (14))$$
$$= \mathrm{Im}\left[\sum_{k=0}^{\infty} \frac{(re^{ix})^{2k}}{(2k)!}\right] \tag{20}$$
$$= \sum_{k=1}^{\infty} \frac{r^{2k}}{(2k)!}\sin 2kx$$

It follows that

$$\int_0^{2\pi} \sinh^2(r\cos x)\cdot\sin^2(r\sin x)dx$$
$$= \pi\cdot\sum_{k=1}^{\infty} \frac{r^{4k}}{[(2k)!]^2}$$

(By (20) and Parseval's theorem)
From the summation of (16) and (17) and using (15), we have

$$\int_0^{2\pi} [\sinh^2(r\cos x)+\cos^2(r\sin x)]dx$$
$$= 2\pi + 2\pi\cdot\sum_{k=1}^{\infty} \frac{r^{4k}}{[(2k)!]^2}$$

3. Examples

In the following, for the six types of definite integrals in this study, we provide some definite integrals and use Theorems 1 and 2 to determine their infinite series forms. On the other hand, we employ Maple to calculate the approximations of these definite integrals and their solutions for verifying our answers.

3.1. Example 1

Taking $r = 7$ into (9), we obtain the definite integral

$$\int_0^{2\pi} \sinh^2(7\cos x)\cdot\cos^2(7\sin x)dx = \pi\cdot\sum_{k=0}^{\infty}\frac{7^{4k+2}}{[(2k+1)!]^2} \quad (21)$$

Next, we use Maple to verify the correctness of (21).

```
>evalf(int((sinh(7*cos(x)))^2*(cos(7*sin(x)))^2,x=0..2*Pi),18);
```

$$2.03289934187665893\cdot 10^5$$

```
>evalf(Pi*sum(7^(4*k+2)/((2*k+1)!)^2,k=0..infinity),18);
```

$$2.03289934187665894\cdot 10^5$$

Also, let $r=\sqrt{3}$ in (10), we have

$$\int_0^{2\pi} \cosh^2(\sqrt{3}\cos x)\cdot\sin^2(\sqrt{3}\sin x)dx = \pi\cdot\sum_{k=0}^{\infty}\frac{(\sqrt{3})^{4k+2}}{[(2k+1)!]^2} \quad (22)$$

```
>evalf(int((cosh(sqrt(3)*cos(x)))^2*(sin(sqrt(3)*sin(x)))^2,x=0..2*Pi),18);
```

11.8342577784367696

```
>evalf(Pi*sum((sqrt(3))^(4*k+2)/((2*k+1)!)^2,k=0..infinity),18);
```

11.8342577784367695

Finally, if $r=5/3$ in (11), then

$$\int_0^{2\pi}\left[\sinh^2\left(\frac{5}{3}\cos x\right)+\sin^2\left(\frac{5}{3}\sin x\right)\right]dx = 2\pi\cdot\sum_{k=0}^{\infty}\frac{(5/3)^{4k+2}}{[(2k+1)!]^2} \quad (23)$$

```
>evalf(int((sinh(5/3*cos(x)))^2+(sin(5/3*sin(x)))^2,x=0..2*Pi),18);
```

21.2666163287006531

```
>evalf(2*Pi*sum((5/3)^(4*k+2)/((2*k+1)!)^2,k=0..infinity),18);
```

21.2666163287006530

3.2. Example 2

Taking $r=9$ into (16), then the definite integral

$$\int_0^{2\pi} \cosh^2(9\cos x)\cdot\cos^2(9\sin x)dx = 2\pi+\pi\sum_{k=1}^{\infty}\frac{9^{4k}}{[(2k)!]^2} \quad (24)$$

```
>evalf(int((cosh(9*cos(x)))^2*(cos(9*sin(x)))^2,x=0..2*Pi),18);
```

$$9.76786250967196351\cdot 10^6$$

```
>evalf(2*Pi+Pi*sum(9^(4*k)/((2*k)!)^2,k=1..infinity),18);
```

$$9.76786250967196350\cdot 10^6$$

In addition, let $r=\sqrt{11}$ in (17) , then

$$\int_0^{2\pi} \sinh^2(\sqrt{11}\cos x)\cdot\sin^2(\sqrt{11}\sin x)dx = \pi\cdot\sum_{k=1}^{\infty}\frac{(\sqrt{11})^{4k}}{[(2k)!]^2} \quad (25)$$

```
>evalf(int((sinh(sqrt(11)*cos(x)))^2*(sin(sqrt(11)*sin(x)))^2,x=0..2*Pi),18);
```

186.043877313544167

```
>evalf(Pi*sum((sqrt(11))^(4*k)/((2*k)!)^2,k=1..infinity),18);
```

186.043877313544167

Finally, if $r=10/7$ in (18), we have

$$\int_0^{2\pi}\left[\sinh^2\left(\frac{10}{7}\cos x\right)+\cos^2\left(\frac{10}{7}\sin x\right)\right]dx = 2\pi+2\pi\cdot\sum_{k=1}^{\infty}\frac{(10/7)^{4k}}{[(2k)!]^2} \quad (26)$$

```
>evalf(int((sinh(10/7*cos(x)))^2+(cos(10/7*sin(x)))^2,x=0..2*Pi),18);
```

13.0155435162085420

```
>evalf(2*Pi+2*Pi*sum((10/7)^(4*k)/((2*k)!)^2,k=1..infinity),18);
```

13.0155435162085419

4. Conclusion

In this paper, we provide a new technique to determine some definite integrals. We hope this technique can be applied to solve another definite integral problems. On the other hand, the Parseval's theorem plays a significant role in the theoretical inferences of this study. In fact, the applications of this theorem are extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. In addition, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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