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Some Relationships between the Generalized Apostol-Bernoulli and Apostol-Euler Polynomials

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Abstract The main objective of this paper is to introduce and investigate two new classes of generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)$ and Apostol-Euler polynomials $\mathcal{E}_n^{[m-1,\alpha]}(x;c,a;\lambda)$. In particular, we obtain addition formula for the new class of the generalized Apostol-Bernoulli polynomials. We also give some recurrence relations and Raabe relations for these polynomials.

Keywords: *Bernoulli polynomials and numbers, Apostol-Bernoulli polynomials, Apostol-Euler polynomials, Generalized Apostol-Bernoulli polynomials*

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1. Introduction, Definitions

Bernoulli polynomials play an important role in various expansions and approximation formulas which are useful both in analytic theory of numbers and the classical and the numerical analysis. These polynomials can be defined by various methods depending on the applications. There are six approaches to the theory of Bernoulli polynomials. We prefer here the definition by generating functions given by Euler [4].

The classical Bernoulli polynomials and the classical Euler polynomials are defined respectively as

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}=\frac{te^{xt}}{e^t-1}, |t|<2\pi, \tag{1.1}$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}=\frac{2e^{xt}}{e^t+1}, |t|<\pi. \tag{1.2}$$

The corresponding Bernoulli numbers B_n and Euler numbers E_n are given by

$$B_n := B_n(0)=(-1)^n B_n(1),$$

$$E_n := 2^n E_n\left(\frac{1}{2}\right), n\in\mathbb{N}_0=\{0\}\cup\{\mathbb{N}\}.$$

From (1.1) and (1.2), we easily derive that

$$B_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k(x),$$

$$E_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}E_k(x),$$

$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k},$$

$$E_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k},$$

$$B_n(x+1)-B_n(x)=nx^{n-1}, n\geq 1, \tag{1.3}$$

$$E_n(x+1)+E_n(x)=2x^n \tag{1.4}$$

(for details, see [11,12,13]).

The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ order $\alpha\in\mathbb{C}$ and the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ order $\alpha\in\mathbb{C}$ are defined respectively by the following generating functions

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}=\left(\frac{t}{\lambda e^t-1}\right)^{\alpha} e^{xt}, \tag{1.5}$$
$$|t+\ln\lambda|<2\pi, 1^{\alpha}:=1,$$

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}=\left(\frac{2}{\lambda e^t+1}\right)^{\alpha} e^{xt}, \tag{1.6}$$
$$|t+\ln\lambda|<\pi, 1^{\alpha}:=1.$$

Recently, Srivastava *et. al.* in ([13,14,15]) have investigated some new classes of Apostol-Bernoulli, Apostol-Euler polynomials with parameters a, b, and c. They gave some recurrence relations and proved some theorems.

For $\lambda=1$ one can obtain the classical polynomials (1.1) and (1.2). Other generalizations can be developed as well.

Definition 1. *[Natalini [12] and S. Chen et al. [3]] The generalized Bernoulli polynomials* $\mathcal{B}_n^{[m-1]}(x), m \geq 1$ *are defined, in a suitable neigbourhood of* $t=0$, *by means of the generating functions*

$$G^{[m-1]}(x,t)=\frac{t^m e^{xt}}{e^t-\sum_{h=0}^{m-1}\frac{t^h}{h!}}=\sum_{n=0}^{\infty}\mathcal{B}_n^{[m-1]}(x)\frac{t^n}{n!}. \quad (1.7)$$

From (1.7) for $m=1$, we obtain the generating function $G^{[0]}(x,t)=\frac{t}{e^t-1}e^{xt}$ of classical Bernoulli polynomials $\mathcal{B}_n^{[0]}(x)$. From (1.7) for $x=0$, we obtain the generalized Bernoulli numbers $\mathcal{B}_n^{[m-1]}$.

Definition 2. *[Kurt [9]] For* $m \in \mathbb{N}$, *the generalized Bernoulli polynomials* $\mathcal{B}_n^{[m-1,\alpha]}(x)$ *of order* $\alpha \in \mathbb{C}, m \in \mathbb{N}$ *are defined by means of the generating function*

$$G^{[m-1,\alpha]}(x,t)=\left(\frac{t^m}{e^t-\sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt} \quad (1.8)$$
$$=\sum_{n=0}^{\infty}\mathcal{B}_n^{[m-1,\alpha]}(x)\frac{t^n}{n!}$$

in suitable neigbourhood of $t=0$.

The case $\alpha=1$ was first introduced by Natalini and Bernardini [6]. For $\alpha=m=1$, we obtain classical Bernoulli polynomials.

By the same motivation, the generalized Euler polynomials $\mathcal{E}_n^{[m-1,\alpha]}(x)$ of order $\alpha \in \mathbb{C}$ and generalized Euler numbers $\mathcal{E}_n^{[m-1,\alpha]}$ of order $\alpha \in \mathbb{C}$ were defined by the author [10]

$$\left(\frac{2^m}{e^t+\sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty}\mathcal{E}_n^{[m-1,\alpha]}(x)\frac{t^n}{n!} \quad (1.9)$$

and

$$\left(\frac{2^m}{e^t+\sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}=\sum_{n=0}^{\infty}\mathcal{E}_n^{[m-1,\alpha]}\frac{t^n}{n!}. \quad (1.10)$$

From (1.9) and (1.10) and for $\alpha=m=1$, we obtain classical Euler polynomials and classical Euler numbers respectively:

$$\mathcal{E}_n^{[0,1]}(x)=E_n(x), 2^n\mathcal{E}_n^{[0,1]}\left(\frac{1}{2}\right)=E_n.$$

By the same motivation, the generalized Genocchi polynomials $\mathcal{G}_n^{[m-1,\alpha]}(x)$ of order $\alpha \in \mathbb{C}$ and generalized Genocchi numbers $\mathcal{G}_n^{[m-1,\alpha]}$ of order $\alpha \in \mathbb{C}$ can be defined as

$$\left(\frac{2^m t^m}{e^t+\sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{xt}=\sum_{n=0}^{\infty}\mathcal{G}_n^{[m-1,\alpha]}(x)\frac{t^n}{n!} \quad (1.11)$$

and

$$\left(\frac{2^m t^m}{e^t+\sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}=\sum_{n=0}^{\infty}\mathcal{G}_n^{[m-1,\alpha]}\frac{t^n}{n!}. \quad (1.12)$$

2. New Classes of Generalized Apostol-Euler Polynomials and Apostol-Bernoulli Polynomials

The following definitions provide a natural generalization of the Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ and the Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;\lambda)$ of order $\alpha \in \mathbb{C}$, where $m \in \mathbb{N}$.

Definition 3. *We de.ne the generalized Bernoulli polynomials* $\mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)$ *of order* $\alpha \in \mathbb{C}$ *and the generalized Euler polynomials* $\mathfrak{E}_n^{[m-1,\alpha]}(x;c,a;\lambda)$ *of order* $\alpha \in \mathbb{C}$ *respectively by*

$$\sum_{n=0}^{\infty}\mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)\frac{t^n}{n!}=\left(\frac{t^m}{\lambda c^t-\sum_{h=0}^{m-1}\frac{(t\ln a)^h}{h!}}\right)^{\alpha} c^{xt}, \quad (2.1)$$

$$\sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\alpha]}(x;c,a;\lambda)\frac{t^n}{n!}=\left(\frac{2^m}{\lambda c^t+\sum_{h=0}^{m-1}\frac{(t\ln a)^h}{h!}}\right)^{\alpha} c^{xt}, \quad (2.2)$$

and

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{[m-1,\alpha]}(x;c,a;\lambda)\frac{t^n}{n!}=\left(\frac{2^m t^m}{\lambda c^t+\sum_{h=0}^{m-1}\frac{(t\ln a)^h}{h!}}\right)^{\alpha} c^{xt}. \quad (2.3)$$

For $\lambda=\alpha=1, c=a=e$, (2.1) reduces to (1.7).

For $\lambda=\alpha=m=1, c=e,$ (2.1), (2.2) and (2.3) reduce to classical Bernoulli polynomial, classical Euler polynomial and classical Genocchi polynomial.

From (2.1), (2.2) and (2.3), we obtain

$$\mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(0;c,a;\lambda)(x\ln c)^k,$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(x;c,a;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_{n-k}^{[m-1,\alpha]}(0;c,a;\lambda)(x\ln c)^k,$$

and

$$\mathcal{G}_n^{[m-1,\alpha]}(x;c,a;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_{n-k}^{[m-1,\alpha]}(0;c,a;\lambda)(x\ln c)^k.$$

Theorem 1. *Let $c\in\mathbb{R}^+,\alpha,\beta\in\mathbb{C},m\in\mathbb{N}$. Then the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)$ and the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;c,a;\lambda)$ satisfy the following relations*

$$\mathfrak{B}_n^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{B}_{n-k}^{[m-1,\alpha]}(x;c,a;\lambda)\mathfrak{B}_{n-k}^{[m-1,\beta]}(y;c,a;\lambda) \tag{2.4}$$

$$\mathfrak{E}_n^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_{n-k}^{[m-1,\alpha]}(x;c,a;\lambda)\mathfrak{E}_{n-k}^{[m-1,\beta]}(y;c,a;\lambda) \tag{2.5}$$

and

$$\mathcal{G}_n^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_{n-k}^{[m-1,\alpha]}(x;c,a;\lambda)\mathcal{G}_{n-k}^{[m-1,\beta]}(y;c,a;\lambda) \tag{2.6}$$

respectively.

Proof. Considering the generating function (2.1) and comparing the coefficients of $\frac{t^n}{n!}$ in the both sides of the above equation, we arrive at (2.4). Proof of (2.5) and (2.6) are similar. □

Theorem 2. *The generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)$ satisfy the following recurrence relation:*

$$\lambda\mathfrak{B}_n^{[m-1,\alpha]}(x+1;c,a;\lambda)-\mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)=n\sum_{k=0}^{n-1}\binom{n-1}{k}\mathfrak{B}_k^{[m-1,\alpha]}(x;c,a;\lambda)\mathfrak{B}_{n-1-k}^{[0,-1]}(0;c,a;\lambda). \tag{2.7}$$

Proof. Considering the expression $\lambda\mathfrak{B}_n^{[m-1,\alpha]}(x+1;c,a;\lambda)-\mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)$ and using generating function (2.1), the proof follows. □

Corollary 1. *The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;c,a;\lambda)$ satisfy the following recurrence relation:*

$$\lambda\mathfrak{E}_n^{[m-1,\alpha]}(x+1;c,a;\lambda)+\mathfrak{E}_n^{[m-1,\alpha]}(x;c,a;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x;c,a;\lambda)\mathfrak{E}_{n-k}^{[0,-1]}(0;c,a;\lambda). \tag{2.8}$$

Theorem 3. *There is the following relation between the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[0,\alpha]}(x;c,a;\lambda)$ for $m=1$ and the generalized Apostol-Euler polynomials $\mathfrak{E}_k^{[0,\alpha]}(x;c,a;\lambda)$ for $m=1$:*

$$\mathfrak{B}_n^{[0,\alpha]}\left(\frac{x+y}{2};c,a;\lambda^2\right)=2^{-n}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[0,\alpha]}\left(\frac{x}{2};c,a;\lambda\right)\mathfrak{B}_{n-k}^{[0,\alpha]}\left(\frac{y}{2};c,a;\lambda\right) \tag{2.9}$$

Proof. We take $m=1;\frac{x+y}{2},\lambda^2$ and $2t$ instead of x,λ and t respectively. We write as:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{B}_n^{[0,\alpha]}\left(\frac{x+y}{2};c,a;\lambda^2\right)\frac{2^nt^n}{n!}\\
&=\left(\frac{2^m}{\lambda c^t+1}\right)^{\alpha}c^{\frac{x}{2}t}\left(\frac{t^m}{\lambda c^t-1}\right)^{\alpha}c^{\frac{y}{2}t}\\
&=\left(\sum_{n=0}^{\infty}\mathfrak{E}_n^{[0,\alpha]}\left(\frac{x}{2};c;\lambda\right)\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\mathfrak{B}_l^{[0,\alpha]}\left(\frac{y}{2};c;\lambda\right)\frac{t^l}{l!}\right)\\
&=\sum_{n=0}^{\infty}\left(2^{-n}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[0,\alpha]}\left(\frac{x}{2};c;\lambda\right)\mathfrak{B}_{n-k}^{[0,\alpha]}\left(\frac{y}{2};c;\lambda\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain (2.9). □

Theorem 4. *The generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x;c,a;\lambda)$ satisfy the following recurrence relation:*

$$\begin{aligned}
&\mathfrak{B}_n^{[0,\alpha-\beta]}(x;c,a;\lambda)\\
&=(-1)^p\frac{n!}{(n+p)!}\left\{\mathfrak{B}_{n+p}^{[0,\alpha]}(x;c,a;\lambda)+\sum_{k=0}^{n+p}\binom{n+p}{k}\sum_{r=0}^{p}\binom{p}{r}\right.\\
&\left.\times(-1)^r\lambda^r\mathfrak{B}_k^{[0,\alpha]}(x;c,a;\lambda)(r\ln c)^{n+p-k}\right\}.
\end{aligned} \tag{2.10}$$

Proof. From (2.1) for $m=1,$ we write as

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathfrak{B}_n^{[0,\alpha-p]}(x;c,a;\lambda)\frac{t^n}{n!}=\left(\frac{t}{\lambda c^t-1}\right)^{\alpha-p}c^{xt}\\
&=\left(\frac{\lambda c^t-1}{t}\right)^p\left(\sum_{n=0}^{\infty}\mathfrak{B}_n^{[0,\alpha]}(x;c,a;\lambda)\frac{t^n}{n!}\right)\\
&t^p\sum_{n=0}^{\infty}\mathfrak{B}_n^{[0,\alpha-p]}(x;c,a;\lambda)\frac{t^n}{n!}\\
&=\left(\lambda c^t-1\right)^p\left(\sum_{k=0}^{\infty}\mathfrak{B}_k^{[0,\alpha]}(x;c,a;\lambda)\frac{t^k}{k!}\right)
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&\left(1-\lambda e^{t\ln c}\right)^p=\sum_{r=0}^{p}\binom{p}{r}(-1)^r\lambda^r\sum_{l=0}^{\infty}(r\ln c)^l\frac{t^l}{l!}\\
&=1+\sum_{l=0}^{\infty}\sum_{r=0}^{p}\binom{p}{r}(-1)^r\lambda^r(r\ln c)^l\frac{t^l}{l!}.
\end{aligned} \tag{2.12}$$

We put (2.12) in the right hand side of (2.11). Then

$$(-1)^p \sum_{n=p}^{\infty} \mathfrak{B}_{n-p}^{[0,\alpha-p]}(x;c,a;\lambda)\frac{t^n}{(n-p)!}$$
$$= \sum_{n=0}^{\infty} \left(\begin{array}{l} \mathfrak{B}_n^{[0,\alpha]}(x;c,a;\lambda) \\ +\sum_{k=0}^{\infty} \binom{n}{k} \sum_{r=0}^{p} \binom{p}{r} (-1)^r \lambda^r \mathfrak{B}_k^{[0,\alpha]}(x;c,a;\lambda)(r\ln c)^{n-k} \end{array} \right) \frac{t^n}{n!}.$$

If we make necessary operations in the last equation and comparing the coefficients of $\frac{t^n}{n!}$, we arrive (2.10). □

Theorem 5. *The following relations hold true:*

$$\mathfrak{E}_n^{[0,-\alpha]}(x;c,a;\lambda)$$
$$= \sum_{k=0}^{\infty} \binom{n}{k} \mathfrak{B}_k^{[0,-\alpha]}(-w;c,a;\lambda^2) 2^k \mathfrak{B}_{n-k}^{[0,\alpha]}(x+2w;c,a;\lambda) \tag{2.13}$$

and

$$\mathfrak{B}_k^{[0,-\alpha]}(x;c,a;\lambda^2)$$
$$= 2^{-n} \sum_{k=0}^{\infty} \binom{n}{k} \mathfrak{B}_k^{[0,\alpha]}(x;c,a;\lambda) \mathfrak{E}_{n-k}^{[0,\alpha]}(x;c,a;\lambda). \tag{2.14}$$

Proof. From (2.1) for $m=1$,

$$\sum_{n=0}^{\infty} \mathfrak{E}_n^{[0,-\alpha]}(x;c,a;\lambda)\frac{t^n}{n!}$$
$$= \left(\frac{2t}{\lambda^2 c^{2t}-1}\right)^{-\alpha} c^{2t(-w)} \left(\frac{t}{\lambda c^t -1}\right)^{\alpha} c^{t(x+2w)}$$
$$= \left(\sum_{k=0}^{\infty} \mathfrak{B}_k^{[0,-\alpha]}(-w;c,a;\lambda^2)\frac{2^k t^k}{k!}\right)$$
$$\times \left(\sum_{l=0}^{\infty} \mathfrak{B}_l^{[0,\alpha]}(x+2w;c,a;\lambda)\frac{t^l}{l!}\right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \begin{array}{l} \binom{n}{k} \mathfrak{B}_k^{[0,-\alpha]}(-w;c,a;\lambda^2) \\ 2^k \mathfrak{B}_{n-k}^{[0,\alpha]}(x+2w;c,a;\lambda) \end{array} \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain (2.13).

For the proof of (2.14), we write

$$\sum_{k=0}^{\infty} \mathfrak{B}_k^{[0,\alpha]}(x;c,a;\lambda^2)\frac{2^n t^n}{n!}$$
$$= \left(\frac{2t}{\lambda^2 c^{2t}-1}\right)^{\alpha} c^{2xt}$$
$$= \left(\frac{t}{\lambda c^t -1}\right)^{\alpha} c^{2xt} \left(\frac{2}{\lambda c^t +1}\right) c^{xt}$$
$$= \sum_{k=0}^{\infty} \left(2^{-n} \sum_{k=0}^{\infty} \binom{n}{k} \mathfrak{B}_k^{[0,\alpha]}(x;c,a;\lambda) \mathfrak{E}_{n-k}^{[0,\alpha]}(x;c,a;\lambda) \right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we arrived to result. □

Corollary 2. *The new generalized Bernoulli polynomials* $\mathfrak{B}_k^{[m-1,\alpha]}(x;c,a)$ *for* $\alpha=m=1$ *and the new generalized Euler polynomials* $\mathfrak{E}_k^{[m-1,\alpha]}(x;c,a)$ *for* $\alpha=m=1$ *satisfy the following Raabe relations:*

$$\sum_{k=0}^{l-1} \mathfrak{B}_n^{[0,1]}\left(\frac{x+k}{l};c,a\right) = l^{1-n}\mathfrak{B}_n^{[0,1]}(x;c,a) \tag{2.15}$$

and

$$\sum_{k=0}^{l-1} \mathfrak{E}_n^{[0,1]}\left(\frac{x+k}{l};c,a\right) = l^{-n}\mathfrak{E}_n^{[0,1]}(x;c,a),\ \textit{for } l \textit{ odd}. \tag{2.16}$$

Proof. We put $m=\lambda=\alpha=1$ in (2.1),

$$\sum_{n=0}^{\infty} \sum_{k=0}^{l-1} \mathfrak{B}_n^{[0,1]}\left(\frac{x+k}{ml};c,a\right)\frac{t^n}{n!}$$
$$= \sum_{k=0}^{l-1} \frac{t}{c^t-1} c^{t\left(\frac{x+k}{l}\right)}$$
$$= \frac{t}{c^t-1} c^{\frac{tx}{l}} \sum_{k=0}^{l-1} e^{\left(\frac{t}{l}\ln c\right)^k}$$
$$= l^{1-n} \sum_{n=0}^{\infty} \mathfrak{B}_n^{[0,1]}(x;c,a)\frac{t^n}{n!}.$$

From the last equality, we have (2.15).

Second equation of this corollary can be obtained similarly, so we omit it. □

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