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Research Article

Bayesian Approximation Techniques of Topp-Leone Distribution

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Topp-Leone distribution is a continuous unimodal distribution used for modeling lifetime phenomena. Topp-Leone distribution has a J-shaped density function with a hazard function of bathtub-shaped and has wide range of applications in reliability fields. This distribution has attracted recent attention for the statistician but has not been discussed in detail in Bayesian approach. This paper is concerned with the estimation of shape parameter of Topp-Leone Distribution using various Bayesian approximation techniques like normal approximation, Tierney and Kadane (T-K) Approximation. Different informative and non-informative priors are used to obtain the Baye's estimate of parameter of Topp-Leone Distributions under different approximation techniques. A simulation study has also been conducted for comparison of Baye's estimates obtained under different approximation using different priors.

Keywords: Bayesian estimation, prior distribution, normal approximation, T-K approximation.

INTRODUCTION

Topp and Leone (1955) defined Topp-Leone distribution and derived the first four moments of the distribution and used it as a model for some failure data. Let X be a random variable following Topp-Leone distribution with probability density function and cumulative distribution function given as

$$f(x) = 2\lambda(1-x)(2x-x^2)^{\lambda-1} \;\; ; 0<x<1; \lambda>0 \qquad (1)$$

$$F(x) = (2x-x^2)^{\lambda}$$

The structural properties of the distribution, an expression for its characteristic function and a bivariate generalization were derived by Nadarajah and Kotz (2003, 2006). Van Dorp and Kotz (2004) used a generalized version of the Topp–Leone distribution called the reflected generalized Topp–Leone distribution and studied its properties. Some reliability measures of the distribution and asymptotic distribution of order statistics were discussed by Ghitany et al. (2005, 2006). Vicari et al. (2008) introduced a two-sided generalized version of the distribution along with some of its properties including moments of the distribution and maximum likelihood method for parameter estimation. Genc (2012) studied the moments of order statistics from Topp–Leone distribution and obtained closed form expressions for single moments and a moment relation. Al-Zahrani (2012) discussed a class of goodness-of-fit tests for the Topp-Leone distribution with estimated parameters.

Feroze et al. (2013) studied Bayesian analysis of failure rate (shape parameter) for Topp-Leone distribution under different loss functions and a couple of non informative priors using singly type II censored samples doubly type II censored samples. Naz Sindhu et al. (2013) studied the problem of estimating the parameter of Topp-Leone distribution based on trimmed samples under informative and non informative priors using Bayesian approach. El-Sayed et al.

(2013) discussed Bayesian and non-Bayesian estimation of Topp-Leone distribution based on lower record values. Mir Mostafaee (2014) presented recurrence relations for the moments of order statistics from Topp-Leone distribution without any restriction for the shape parameter.

MATERIAL AND METHODS

The Bayesian paradigm is conceptually simple and probabilistically elegant. Sometimes posterior distribution is expressible in terms of complicated analytical function and requires intensive calculation because of its numerical implementations. It is therefore useful to study approximate and large sample behavior of posterior distribution. Thus, our present study focuses to obtain the estimates of shape parameter of Topp-Leone distribution using two Bayesian approximation techniques i.e. normal approximation, T-K approximation.

Normal Approximation:

The basic result of the large sample Bayesian inference is that the posterior distribution of the parameter approaches a normal distribution. If the posterior distribution $P(\lambda \mid x)$ is unimodal and roughly symmetric, it is convenient to approximate it by a normal distribution centered at the mode; that is logarithm of the posterior is approximated by a quadratic function, yielding the approximation

$$P(\lambda \mid x) \sim N\left(\hat{\lambda}, \left[I(\hat{\lambda})\right]^{-1}\right)$$

where $I(\hat{\lambda}) = -\dfrac{\partial^2 \log P(\lambda \mid y)}{\partial \lambda^2}$ (2)

If the mode, $\hat{\lambda}$ is in the interior parameter space, then $I(\lambda)$ is positive; if $\hat{\lambda}$ is a vector parameter, then $I(\lambda)$ is a matrix.

A review on this topic is discussed by Freedman, Spiegel halter and Parmer (1994), Khan, A.A (1997) and Khan et al. (1996). Ahmad et al. (2007, 2011) discussed Bayesian analysis of exponential distribution and gamma distribution using normal and Laplace approximations.

In our study the normal approximations of Topp-Leone distribution under different priors is obtained as under:

The likelihood function of (1) for a sample of size n is given as

$$L(\underline{x} \mid \lambda) \propto (\lambda)^n \, e^{-\lambda \sum_{i=1}^{n} \ln(2x_i - x_i^2)^{-1}} \tag{3}$$

Under uniform prior $g(\lambda) \propto 1$, the posterior distribution for λ is as

$$P(\lambda \mid x) \propto \lambda^n e^{-\lambda S} \quad \text{where } S = \sum_{i=1}^{n} \ln(2x_i - x_i^2)^{-1} \tag{4}$$

Log posterior is $\ln P(\lambda \mid x) = \ln \text{constant} + n \ln \lambda - \lambda S$

Therefore first derivative is $\dfrac{\partial \ln P(\lambda \mid x)}{\partial \lambda} = \dfrac{n}{\lambda} - S$

from which the posterior mode is obtained as $\hat{\lambda} = \dfrac{n}{S}$

and the negative of Hessian $I(\hat{\lambda}) = -\dfrac{\partial^2 \ln P(\lambda \mid x)}{\partial \lambda^2} = \dfrac{S^2}{n}$

$$\left[I(\hat{\lambda})\right]^{-1} = \frac{n}{S^2}$$

Thus, the posterior distribution can be approximated as

$$P(\lambda \mid x) \sim N\left(\frac{n}{S}; \frac{n}{S^2}\right) \tag{5}$$

Under extension of Jeffrey's prior $g(\lambda) \propto \left(\frac{1}{\lambda}\right)^m ; m \in R^+$, the posterior distribution for λ is as

$$P(\lambda \mid x) \propto \lambda^{n-m} e^{-\lambda S} \tag{6}$$

Log posterior $\ln P(\lambda \mid x) = \ln \text{constant} + (n-m)\ln\lambda - \lambda S$

The first derivative is $\frac{\partial \ln P(\lambda \mid x)}{\partial \lambda} = \frac{n-m}{\lambda} - S$

from which the posterior mode is obtained as $\hat{\lambda} = \frac{n-m}{S}$

Therefore, $\left[I(\hat{\theta})\right]^{-1} = \frac{n-m}{S^2}$

Thus, the posterior distribution can be approximated as

$$P(\lambda \mid x) \sim N\left(\frac{n-m}{S}; \frac{n-m}{S^2}\right) \tag{7}$$

Under gamma prior $g(\lambda) \propto e^{-a\lambda} \lambda^{b-1}$; $a, b > 0; \lambda > 0$ where a, b are the known hyper parameters. The posterior distribution for λ is as

$$P(\lambda \mid x) \propto \lambda^{n+b-1} e^{-\lambda(a+S)} \tag{8}$$

Log posterior is $\ln P(\lambda \mid x) = \ln \text{constant} + (n+b-1)\ln\lambda - \lambda(a+S)$

$\therefore$ posteriormode $\hat{\lambda} = \frac{n+b-1}{(a+S)}$

and $\left[I(\hat{\lambda})\right]^{-1} = \frac{n+b-1}{[S+a]^2}$

Thus, the posterior distribution can be approximated as

$$P(\lambda \mid x) \sim N\left(\frac{n+b-1}{S+a}; \frac{n+b-1}{[S+a]^2}\right) \tag{9}$$

Under the Exponential prior $g(\lambda) \propto c e^{-\lambda c}; c > 0; \lambda > 0$, where c is the known hyper parameter, the posterior distribution for λ is as

$$P(\lambda \mid x) \propto \lambda^n e^{-\lambda(c+S)} \tag{10}$$

$\therefore$ posteriormode $\hat{\lambda} = \frac{n}{(c+S)}$

and $\left[I(\hat{\lambda})\right]^{-1} = \frac{n}{[S+c]^2}$

Thus, the posterior distribution can be approximated as

$$P(\lambda \mid x) \sim N\left(\frac{n}{S+c}; \frac{n}{[S+c]^2}\right) \tag{11}$$

T-K Approximation:

In Lindley's approximation one requires the evaluation of third order partial derivatives of likelihood function which may be cumbersome to compute when the parameter λ is a vector valued parameter thus, Tierney and Kadane (1986) gave Laplace method to evaluate $E(h(\lambda) \mid x)$ as

$$E(h(\lambda)\,|\,x) \cong \frac{\hat{\phi}^*}{\hat{\phi}} \frac{\exp\{-n\,h''^*(\hat{\lambda}^*)\}}{\exp\{-n\,h''(\hat{\lambda})\}} \tag{12}$$

where $-n\,h''(\hat{\lambda}) = \ln P(\lambda\,|\,x)$; $-n\,h''^*(\hat{\lambda}^*) = \ln P(\lambda\,|\,x) + \ln h(\lambda)$;

$$\hat{\phi}^2 = -\left[-n\,h''(\hat{\lambda})\right]^{-1};\ \hat{\phi}^{*2} = -\left[-n\,h''^*(\hat{\lambda}^*)\right]^{-1}$$

Thus, for Topp-Leone Distribution Laplace approximation for shape parameter λ can be calculated as Under uniform prior $g(\lambda) \propto 1$, the posterior distribution for λ is given in (4)

$$\therefore\ -n\,h(\lambda) = n\ln\lambda - \lambda S\ ;\ -n\,h'(\lambda) = \frac{n}{\lambda} - S\ ;$$

$$\Rightarrow \hat{\lambda} = \frac{n}{S}$$

Also $-n\,h''(\hat{\lambda}) = -\dfrac{S^2}{n}$

Therefore $\hat{\phi}^2 = -\left[-n\,h''(\hat{\lambda})\right]^{-1} = \dfrac{n}{S^2}$ or $\hat{\phi} = \dfrac{(n)^{1/2}}{S}$

now $-n\,h^*(\lambda^*) = -n\,h(\lambda) + \ln h(\lambda) = (n+1)\ln\lambda^* - \lambda^* S$

$$-n\,h'^*(\lambda^*) = \frac{n+1}{\lambda^*} - S\ \Rightarrow \hat{\lambda}^* = \frac{n+1}{S}$$

Also $-n\,h''^*(\hat{\lambda}^*) = -\dfrac{S^2}{n+1}\ \Rightarrow \hat{\phi}^* = \dfrac{(n+1)^{1/2}}{S}$

Thus using (12) we have

$$E(\lambda\,|\,x) = \frac{(n+1)^{1/2}}{(n)^{1/2}} \frac{\exp\{(n+1)\ln\hat{\lambda}^* - \hat{\lambda}\,S\}}{\exp\{n\ln\hat{\lambda} - \hat{\lambda}\,S\}} = \left(\frac{n+1}{n}\right)^{1/2} \frac{\hat{\lambda}^{*n+1}}{\hat{\lambda}^n} e^{-\hat{\lambda}^* S + \hat{\lambda} S}$$

$$= \frac{n+1}{S}\left(\frac{n+1}{n}\right)^{n+1/2} e^{-1} \tag{13}$$

Note that the relative error (relative error to exact the posterior mean $\dfrac{n+1}{S}$) is $\left(\dfrac{n+1}{n}\right)^{n+1/2} e^{-1}$

Similarly $E(\lambda^2\,|\,x) = \dfrac{\hat{\phi}^*}{\hat{\phi}} \dfrac{\exp\{-n\,h''^*(\hat{\lambda}^*)\}}{\exp\{-n\,h''(\hat{\lambda})\}}; here\ -n\,h''^*(\hat{\lambda}^*) = \log(\lambda^2) - n\,h(\lambda)$

$$E(\lambda^2\,|\,x) = \left(\frac{n+1}{n}\right)^{n+1/2} \left(\frac{n+2}{S}\right)^2 e^{-2}$$

$$\therefore Variance = \left(\frac{n+1}{n}\right)^{n+1/2} \left(\frac{n+2}{S}\right)^2 e^{-2} - \left[\left(\frac{n+1}{S}\right)\left(\frac{n+1}{n}\right)^{n+1/2} e^{-1}\right]^2$$

Under extension of Jeffrey's prior $g(\lambda) \propto \left(\dfrac{1}{\lambda}\right)^m, m \in R^+$ the posterior distribution for λ is given in (6)

$$\therefore -n\,h(\lambda) = (n-m)\ln\lambda - \lambda S\ ;\ -n\,h'(\lambda) = \frac{n-m}{\lambda} - S\ \Rightarrow \hat{\lambda} = \frac{n-m}{S}$$

Also $-n h''(\hat{\lambda}) = -\dfrac{S^2}{n-m} \Rightarrow \hat{\phi} = \dfrac{(n-m)^{1/2}}{S}$

now $-n h^*(\lambda^*) = -n h(\lambda) + \ln h(\lambda) = (n-m+1)\ln\lambda^* - \lambda^* S$

$\therefore -n h'^*(\lambda^*) = \dfrac{n-m+1}{\lambda^*} - S \Rightarrow \hat{\lambda}^* = \dfrac{n-m+1}{S}$

and $-n h''^*(\hat{\lambda}^*) = -\dfrac{S^2}{n-m+1} \Rightarrow \hat{\phi}^* = \dfrac{(n-m+1)^{1/2}}{S}$

Thus $E(\lambda \mid x) = \left(\dfrac{n-m+1}{n-m}\right)^{1/2} \dfrac{\exp\{(n-m+1)\ln\hat{\lambda}^* - \hat{\lambda}^* S\}}{\exp\{(n-m)\ln\lambda - \lambda S\}} = \left(\dfrac{n-m+1}{n-m}\right)^{1/2} \dfrac{\hat{\lambda}^{*n-m+1}}{\hat{\lambda}^{n-m}} e^{-\hat{\lambda}^* S + \hat{\lambda} S}$

$$= \left(\frac{n-m+1}{S}\right)\left(\frac{n-m+1}{n-m}\right)^{n-m+1/2} e^{-1} \tag{14}$$

Note that the relative error (relative error to exact the posterior mean $\dfrac{n-m+1}{S}$) is $\left(\dfrac{n-m+1}{n-m}\right)^{n-m+1/2} e^{-1}$.

Further $E(\lambda^2 \mid x) = \left(\dfrac{n-m+1}{S}\right)^2 \left(\dfrac{n-m+1}{n-m}\right)^{n-m+1/2} e^{-2}$

$\therefore Variance = \left(\dfrac{n-m+1}{S}\right)^2 \left(\dfrac{n-m+1}{n-m}\right)^{n-m+1/2} e^{-2} - \left[\left(\dfrac{n-m+1}{S}\right)\left(\dfrac{n-m+1}{n-m}\right)^{n-m+1/2} e^{-1}\right]^2$

Under Gamma prior $g(\lambda) \propto e^{-a\lambda}\lambda^{b-1}$; the posterior distribution for λ is given in (8)

$\therefore -n h(\lambda) = (n+b-1)\ln\lambda - \lambda(S+a)$; $-n h'(\theta) = \dfrac{n+b-1}{\lambda} - (S+a)$

$\Rightarrow \hat{\lambda} = \dfrac{n+b-1}{(S+a)}$

Also $-n h''(\lambda) = -\dfrac{(S+a)^2}{(n+b-1)} \Rightarrow \hat{\phi} = \dfrac{(n+b-1)^{1/2}}{(S+a)}$

now $-n h^*(\theta^*) = -n h(\theta) + \ln h(\theta) = (n+b)\ln\lambda^* - (S+a)\lambda^*$

$-n h'^*(\lambda^*) = \dfrac{(n+b)}{\lambda^*} - (S+a) \therefore \hat{\lambda}^* = \dfrac{(n+b)}{(S+a)}$

also $-n h''^*(\hat{\theta}^*) = -\dfrac{(S+a)^2}{(n+b)} \Rightarrow \hat{\phi}^* = \dfrac{(n+b)^{1/2}}{(S+a)}$

Thus $E(\lambda \mid x) = \left(\dfrac{n+b}{n+b-1}\right)^{1/2} \dfrac{\exp\{(n+b)\ln\hat{\lambda}^* - \hat{\lambda}^*(S+a)\}}{\exp\{(n+b-1)\ln\lambda - \lambda(S+a)\}}$

$$= \left(\frac{n+b}{S+a}\right)\left(\frac{n+b}{n+b-1}\right)^{n+b-1/2} e^{-1} \tag{15}$$

Note that the relative error (relative error to exact the posterior mean $\dfrac{n+b}{S+a}$) is $\left(\dfrac{n+b}{n+b-1}\right)^{n+b-1/2} e^{-1}$

Further $E(\lambda^2 \mid x) = \left(\frac{n+b+1}{S+a}\right)\left(\frac{n+b-1}{S+a}\right)\left(\frac{n+b}{n+b-1}\right)^{n+b+1/2} e^{-2}$

$$\therefore Variance = \left(\frac{n+b+1}{S+a}\right)\left(\frac{n+b-1}{S+a}\right)\left(\frac{n+b}{n+b-1}\right)^{n+b+1/2} e^{-2} - \left[\left(\frac{n+b}{S+a}\right)\left(\frac{n+b}{n+b-1}\right)^{n+b-1/2} e^{-1}\right]^2$$

Under Exponential prior $g(\lambda) \propto c\, e^{-\lambda c}$; the posterior distribution for λ is given in (10)

$$\therefore -n\,h(\lambda) = -(S+c)\lambda + n\ln\lambda \;; -n\,h'(\theta) = \frac{n}{\lambda} - (S+c)$$

$$\Rightarrow \hat{\lambda} = \frac{n}{(S+c)}$$

also $-n\,h''(\hat{\lambda}) = -\frac{(S+c)^2}{n} \;\Rightarrow \hat{\phi} = \frac{(n)^{1/2}}{(S+c)}$

now $-n\,h^{*}(\lambda^*) = -n\,h(\lambda) + \ln h(\lambda) = (n+1)\ln\lambda^* - (S+c)\lambda^*$

$$\Rightarrow -n\,h'^{*}(\lambda^*) = \frac{(n+1)}{\lambda^*} - (S+c) \quad \therefore \hat{\lambda}^* = \frac{(n+1)}{(S+c)}$$

also $-n\,h''^{*}(\hat{\lambda}^*) = -\frac{(S+c)^2}{(n+1)} \;\Rightarrow \hat{\phi}^* = \frac{(n+1)^{1/2}}{(S+c)}$

Therefore $E(\lambda \mid x) = \left(\frac{n+1}{n}\right)^{1/2} \frac{\exp\{(n+1)\ln\hat{\lambda}^* + \hat{\lambda}^*(S+c)\}}{\exp\{n\ln\hat{\lambda} + \hat{\lambda}(S+c)\}}$

$$= \left(\frac{n+1}{S+c}\right)\left(\frac{n+1}{n}\right)^{n+1/2} e^{-1} \qquad (16)$$

Note that the relative error (relative error to exact the posterior mean $\frac{n+1}{S+c}$) is $\left(\frac{n+1}{n}\right)^{n+1/2} e^{-1}$

Also $E(\lambda^2 \mid x) = \left(\frac{n+2}{S+c}\right)^2 \left(\frac{n+2}{n}\right)^{n+1/2} e^{-2}$

$$\therefore Variance = \left(\frac{n+2}{S+c}\right)^2 \left(\frac{n+2}{n}\right)^{n+1/2} e^{-2} - \left[\left(\frac{n+1}{S+c}\right)\left(\frac{n+1}{n}\right)^{n+1/2} e^{-1}\right]^2$$

Simulation study

In order to compare and examine the performance of Bayesian estimates for shape parameter of Topp-Leone distribution under different priors by using different approximation techniques, a simulation study is conducted in R software by using vaRES package. For this purpose a sample of size n=10, 20, 30, 40 and 50 has been generated to observe the effect of small and large samples on the estimators and the results are presented in the below tables.

Table 1. Posterior estimates and posterior variance of Topp-Leone distribution under normal approximation

n	Posterior mean						Posterior variance					
	Uniform prior	**Jeffrey's prior** m=0.5	m=1	m=1.5	**Gamma prior**	**Exponential prior**	**Uniform prior**	**Jeffrey's prior** m=0.5	m=1	m=1.5	**Gamma prior**	**Exponential prior**
10	3.8804	3.6864	3.4924	3.2984	2.5754	2.4527	1.5058	1.4305	1.3552	1.2799	0.6316	0.6016
20	1.0244	0.9988	0.9732	0.9476	0.9751	0.9513	0.0524	0.0511	0.0498	0.0485	0.0463	0.0452
30	1.6172	1.5902	1.5633	1.5363	1.5211	1.4962	0.0871	0.0857	0.0842	0.0828	0.0758	0.0746
40	1.9611	1.9365	1.9120	1.8875	1.8496	1.8267	0.0961	0.0949	0.0937	0.0925	0.0844	0.0834
50	1.7912	1.7733	1.7554	1.7375	1.7169	1.6999	0.0641	0.0635	0.0628	0.0622	0.0583	0.0577

Table 2. Posterior estimates and posterior variance of Topp-Leone distribution under T-K approximation

n	Posterior mean						Posterior variance					
	Uniform prior	**Jeffrey's prior** m=0.5	m=1	m=1.5	**Gamma prior**	**Exponential prior**	**Uniform prior**	**Jeffrey's prior** m=0.5	m=1	m=1.5	**Gamma prior**	**Exponential prior**
10	4.2717	4.0779	3.8840	3.6902	1.1365	2.7001	1.6564	1.5811	1.5058	1.4305	1.8920	0.6617
20	1.0758	1.0502	1.0246	0.9990	0.3945	0.9991	0.0550	0.0537	0.0524	0.0511	0.2470	0.0475
30	1.6712	1.6443	1.6173	1.5904	0.5969	1.5462	0.0901	0.0886	0.0871	0.0857	0.5805	0.0771
40	2.0102	1.9857	1.9612	1.9367	0.7144	1.8725	0.0985	0.0973	0.0961	0.0949	0.8429	0.0855
50	1.8271	1.8092	1.7913	1.7734	0.6568	1.7339	0.0654	0.0648	0.0641	0.0635	0.7182	0.0589

DISCUSSION

In this paper the focus was to study the importance of Bayesian approximation techniques. We presented approximate to Bayesian integrals of Topp-Leone distribution depending upon numerical integration and simulation study and showed how to study posterior distribution by means of simulation study. We observe that under informative as well as non-informative priors, the normal approximation behaves well than T-K approximation, although the posterior variances in case of T-K approximation are very close to that of normal approximation.

CONCLUSION

From the findings of above tables it can be observed that the large sample distribution could be improved when prior is taken into account. In both cases normal approximation as well as Laplace approximation, Bayesian estimates under informative priors are better than those under non-informative priors especially the exponential distribution proves to be efficient with minimum posterior variance with preceding gamma prior in normal approximation. Further in case of non-informative priors Jeffrey's prior for m=1.5 (also known as Hartigan prior) proves to be efficient.

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