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STRONG DOMINATION IN PERMUTATION

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ABSTRACT. Adin and Roichman introduced the concept of permutation graphs and Peter Keevash, Po-Shen Loh and Benny Sudakov identified some permutation graphs with maximum number of edges. Charles J Colbourn, Lorna K.Stewart characterized the connected domination and Steiner Trees under the Permutation graphs. If i, j belong to a permutation π on p symbols $A = \{1, 2, , p\}$ and $i < j$ then the line of i crosses the line of j in the permutation if i appears after j in the image sequence $s(\pi)$ and if the no. of crossing lines of i is less than the no. of crossing lines of j then i strongly dominates j. A subset D of A, whose closed neighborhood is A in π is a dominating set of π. D is a strong dominating set of π if every i in $A - D$ is strongly dominated by some j in D. In this paper the strong number of a permutation is investigated by means of crossing lines.

1. Permutation Graphs

DEFINITION 1.1. Let π be a permutation on a finite set A $= \{a_1, a_2, a_3, \ldots, a_p\}$ given by $\pi = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_p \\ a'_1 & a'_2 & a'_3 & \ldots & a'_p \end{pmatrix}$ where $|a_{i+1} - a_i| = c$, $c > 0$, $0 < i \leqslant p-1$. The sequence of π is given by $s(\pi) = \{a'_1, a'_2, a'_3, \ldots, a'_p\}$.

When elements of A are ordered in L_1 and the sequence of π are represented in L_2, then a line joining a_i in L_1 and a_i in L_2 is represented by l_i. This is known as line representation of a_i in π.

EXAMPLE 1.1. Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$.
Then the line l_1 crosses l_3 and l_5; l_2 crosses l_3 , l_4 and l_5; l_3 crosses l_1 and l_2; l_4 crosses l_2 and l_5; l_5 crosses l_1, l_2 and l_4.

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DEFINITION 1.2. Let $a_i, a_j \in A$. Then the residue of a_i and a_j in π is denoted by Res(a_i, a_j) and is given by $(a_i - a_j)(\pi^{-1}(a_i) - \pi^{-1}(a_j))$.

DEFINITION 1.3. Let l_i and l_j denote the lines corresponding to the elements a_i and a_j respectively. Then l_i crosses l_j if Res $(a_i, a_j) < 0$. If l_i crosses l_j then $(a_i, a_j) \in E_\pi$.

DEFINITION 1.4. Let π be a permutation on a finite set $A = \{a_1, a_2, a_3, \ldots, a_p\}$ given by $\pi = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_p \\ a'_1 & a'_2 & a'_3 & \ldots & a'_p \end{pmatrix}$ where $|a_{i+1} - a_i| = c$, $c > 0$, $0 < i \leqslant p-1$. Then the π-Permutation Graph G_π is given by $G_\pi = (V_\pi, E_\pi)$ where $V_\pi = \{a_1, a_2, \ldots, a_p\}$ and $a_i a_j \in E_\pi$, if Res$(a_i, a_j) < 0$.

LEMMA 1.1. *Let π be a permutation on a finite set $A = \{a_1, a_2, a_3, ..., a_p\}$ given by $\pi = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_p \\ a'_1 & a'_2 & a'_3 & \ldots & a'_p \end{pmatrix}$ where $|a_{i+1} - a_i| = c$, $c > 0$, $0 < i \leqslant p-1$. Then there exists a $1-1$ correspondence between crossing of lines in π and elements of E_π.*

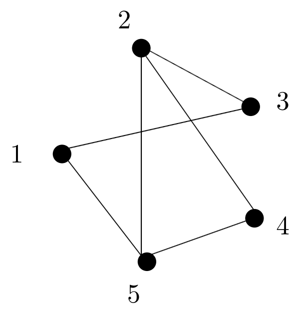
PROOF. Let there be a_i , $a_j \in A$ such that l_i intersects l_j in π. Let us assume $a_i < a_j$. (i.e) $a_i - a_j < 0$.
As l_i intersects l_j, then a_j appears before a_i in $s(\pi)$. (i.e) $(\pi^{-1}(a_i) - \pi^{-1}(a_j)) > 0$.
Hence Res $(a_i, a_j) < 0$ which implies $a_i a_j \in E_\pi$.
Conversely let $a_i a_j \in E_\pi$. (i.e) Res $(a_i, a_j) < 0$. By assumption $a_i - a_j < 0$. Hence $(\pi^{-1}(a_i) - \pi^{-1}(a_j)) > 0$ (i.e) a_j appears before a_i in $s(\pi)$. Hence l_i intersects l_j. □

LEMMA 1.2. *Let π be a permutation on a finite set $A = \{a_1, a_2, a_3, ..., a_p\}$, where $|a_{i+1} - a_i| = c$, $c > 0$, $0 < i \leqslant p-1$. Then Res$(a_i, a_j) =$ Res(a_j, a_i).*

PROOF. Let $a_i - a_j =$ mk, $m \neq 0$.
Let $\pi^{-1}(a_i) = a_r$ and $\pi^{-1}(a_j) = a_s$.
Then $a_r - a_s =$ nk, $n \neq 0$.
Res $(a_i, a_j) = (a_i - a_j)(\pi^{-1}(a_i) - \pi^{-1}(a_j)) =$ mk nk $=$ mn k^2.
Res $(a_j, a_i) = (a_j - a_i)(\pi^{-1}(a_j) - \pi^{-1}(a_i)) =$(- n)k (-m)k $=$ mn k^2.
Hence Res $(a_i, a_j) =$ Res (a_j, a_i). □

DEFINITION 1.5. [**1**] A graph G is a permutation graph if there exists π such that $G_\pi \cong G$. (i.e) a graph is a permutation graph if it is realisable by a permutation π. Otherwise it is not a permutation graph.

EXAMPLE 1.2. Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$. Then $G_\pi = (V_\pi, E_\pi)$ where $V_\pi = \{1, 2, 3, 4, 5\}$ and $E_\pi = \{(1, 3), (1, 5), (2, 3), (2, 4), (2, 5), (4, 5)\}$.



Note 1: [**3**] C_n , $n \geqslant 5$ are not realisable by means of permutations.

Definition 1.6. The neighbourhood of a_i in π is a set of all elements of π whose lines cross the line of a_i and is denoted by $N_\pi(a_i)$, equal to $\{a_r \in \pi/l_i$ crosses $l_r$in $\pi\}$ and $d(a_i) = |N_\pi(a_i)|$ is the number of lines that cross l_i in π.

Definition 1.7. $N_\pi(S)$,neighbourhood of a subset S of V in $\pi = \cup_{a_i \in S} N_\pi(a_i)$ = set of all elements of π whose lines cross the lines of all $a_i \in S$.
The closed neighbourhood of a subset S of V in π is $N_\pi[S]$= $N_\pi(S) \cup S$.

The neighbourhood of a_i in S is a set of all elements of S whose lines cross the line of a_i and is denoted by $N_S(a_i)$, equal to $\{a_r \in S/l_i$ crosses $l_r$in $S\}$

Definition 1.8. Let A be a subset of V. Then $< A > = \cup_{a_i \in A} N_A(a_i) = \{a_i \in A/l_i$ crosses $l_j, a_j \in A\}$. If $< A > = \phi$ then we say that A has trivial crossing. (i.e) for a_r, a_s in A, l_r, l_s do not cross in π.

Example 1.3. Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}$. Here V $=\{1, 2, 3, 4, 5\}$, $N_\pi(1) = \{3, 5\}$; $N_\pi(2) = \{3, 4, 5\}$;$N_\pi(3) = \{1, 2\}$; $N_\pi(4) = \{2, 5\}$; $N_\pi(5) = \{1, 2, 4\}$.
Let S$=\{4, 5\}$. Then $N_\pi(S) = \{1, 2, 4, 5\}$. $N_S(4) = \{5\}$ and $< S >= \{4, 5\}$
Let A $= \{1, 2\}$. Then $< A >= \phi$ and $N_\pi[A] = V$.

2. Domination of a Permutation

Definition 2.1. Let $\pi = \begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_p \\ a'_1 & a'_2 & a'_3 & \ldots & a'_p \end{pmatrix}$. Then a_i is said to dominate a_j if l_i and l_j cross each other in π (may also be trivial). (If it is trivial, then a_i dominates a_i and a_j dominates a_j itself.)

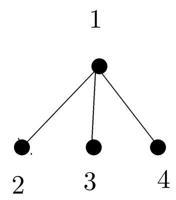
Definition 2.2. The subset D of V$= \{a_1, a_2, \ldots, a_p\}\, in\, \pi$ is said to be a dominating set of π if $N_\pi[D] = V$. V $= \{a_1, a_2, \ldots, a_p\}$ is always a dominating set.

Definition 2.3. The subset D of $\{a_1, a_2, \ldots, a_p\}$ is said to be a minimal dominating set of π, MDS(π), if $D - \{a_j\}$ is not a dominating set of π for all $a_j \in D$. That is D is 1-minimal.

Definition 2.4. [**2**] The domination number of a permutation π is the minimum cardinality of a set of all MDS(π) and is denoted by $\gamma(\pi)$.

EXAMPLE 2.1. Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$. Then $G_\pi = (V_\pi, E_\pi)$ where $V_\pi = \{1, 2, 3, 4\}$ and $E_\pi = \{(1, 2), (1, 3), (1, 4)\}$.

G_π :



Here $D_1 = \{1\}\, and\, D_2 = \{2, 3, 4\}$. Both $D_1 and\, D_2$ are minimal dominating sets of π.

THEOREM 2.1. [**3**] *The domination number of a permutation π is $\gamma(\pi) = \gamma(G_\pi)$, the minimum cardinality of the minimal dominating sets of G_π.*

3. Strong Domination Number of a Permutation

DEFINITION 3.1. Let $\mathrm{Res}(a_i, a_j) < 0$ and let $d(a_i) \geqslant d(a_j)$ then we say a_i strongly dominates a_j and a_j weakly dominates a_i.

DEFINITION 3.2. A subset D of V(π) is said to be a strong dominating set of π if $N_\pi[D] = V(\pi)$ and $d(a_i) \geqslant d(a_j)$ such that for atleast one $a_i \in D$, $a_j \in V(\pi) - D$, Res $(a_i, a_j) < 0$.

DEFINITION 3.3. The subset D of V(π) is said to be a minimal strong dominating set D, MSDS(π), if $D - \{a_j\}$ is not a strong dominating set of π for all $a_j \in D$.

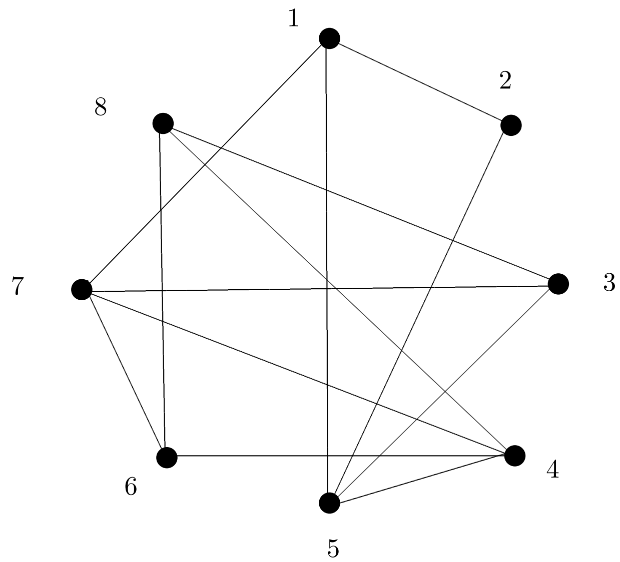
DEFINITION 3.4. The strong domination number of π, is denoted by $\gamma_s(\pi)$ which is the minimum cardinality of all minimal strong dominating sets of π.

THEOREM 3.1. *The strong domination number of a permutation π is $\gamma_s(\pi) = \gamma_s(G_\pi)$, the minimum cardinality of the minimal strong dominating sets ($MSDS$) of G_π.*

PROOF. Let π be a permutation on a finite set V= $\{a_1, a_2, a_3, ..., a_p\}$ given by $\pi = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_p \\ a'_1 & a'_2 & a'_3 & \dots & a'_p \end{pmatrix}$. Let $G_\pi = (V_\pi, E_\pi)$ where $V_\pi =$ V and $a_i a_j \in E_\pi$, if $Res(a_i, a_j) < 0$.
Let $a_i \in V$ such that $d(a_i) =$ max $\{d(a_j)/a_j \in V\}$.
Then D=$\{a_i\}$ and let T = $N_\pi(a_i)$.
Let $V_1 = V - (D \cup T)$.
If there exists only one such a_i and if $V_1 = \phi$, then D is MSDS(π).
If $V_1 \neq \phi$, and $< V_1 >= \phi$ then $D_1 = D \cup V_1$ is a MSDS(π).
If $V_1 \neq \phi$, and $< V_1 > \neq \phi$ then choose $a_r \in V - D$ such that
$d(a_r) =$ max $\{d(a_i)/a_i \in V_1\}$.
If $d(a_r) > d(a_i) \forall a_i \in N_\pi(a_r)$ then $D_1 = D \cup \{a_r\}$ and $T_1 = N_\pi(a_r)$ and
$V_2 = V_1 - (D_1 \cup T_1)$

Otherwise choose $a_t \in N_\pi(a_r)$ such that $d(a_t) = max\{d(a_i)/a_i \in N_\pi(a_r)\}$.
Now $D_1 = D \cup \{a_t\}$ and $T_1 = N_\pi(a_t)$ and $V_2 = V_1 - (D_1 \cup T_1)$.
If $V_2 = \phi$, then D_1 is MSDS(π).
If $V_2 \neq \phi$, and $< V_2 >_\pi = \phi$ then $D_2 = D_1 \cup V_1$ is a MSDS(π).
If $V_2 \neq \phi$, and $< V_2 >_\pi \neq \phi$, then proceed as before to obtain a MSDS.
If there are more than one a_i such $d(a_i)$ is max then by applying the same procedure to all $a_{r_1}, a_{r_2}, \cdots, a_{r_m}$ where $0 \leqslant r_1, r_2, \cdots, r_m \leqslant n$ all MSDS(π) are obtained.
V is finite and no. of subsets of E_π is finite. Hence within 2^n approaches all minimal strong dominating sets including minimum strong dominating set are produced. The minimum cardinality of the sets in all MSDS(π) is the strong domination number of π which is $\gamma_s(\pi)$. So calculation of $\gamma_s(\pi)$ is of polynomial time.
Hence by Lemma 2. 1, $\gamma_s(\pi)$=$\gamma_s(G_\pi)$. □

Example 3.1. Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 2 & 7 & 1 & 8 & 3 & 6 & 4 \end{pmatrix}$.
Here $D_1 = \{4, 5\}\, and\, D_2 = \{1, 4, 7\}$.
Both $D_1 and\, D_2$ are minimal strong dominating sets.
$\gamma_s(\pi) = \gamma_s(G_\pi)$ =2.



Note 2: If either $\pi(a_1) = a_p$ or $\pi(a_p) = a_1$, then l_1 crosses all l_i, $1 < i \leqslant p$, or l_p crosses all l_j, $1 \leqslant i < p$. In both cases G_π has atleast one full degree vertex. Hence $\gamma_s(\pi) = \gamma_s(G_\pi) = 1$.
Note 3: An example for a permutation graph for which $\gamma_s(\pi) = \gamma_s(G_\pi) = i(G_\pi)$ is C_4 and $K_{2,r}$, r-finite.

4. Conclusion

The permutation graphs in terms of crossing of lines and the sequence of permutations were defined and methods of arriving at a dominating set in permutations were discussed by us. The procedure was extended to find a strong dominating set in a permutation graph in this paper. Similarly independent dominating set, minimal independent dominating set and independent domination number of a permutation, $i(\pi)$, can be defined. Hence the domination number, strong domination number and independent domination number of the permutations realising a some standard graphs were found by means of crossing of lines.

References

[1] Frank Harary, *Graph Theory*, Narosa Publishing House, Calcutta, 2001

[2] Teresa W. Haynes, Stephen T. Hedetneimi, Peter J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker,INC., New York 1998

[3] J. Chithra, S. P. Subbiah and V. Swaminathan, Domination in Permutation Graph, *Intenational Journal of Computing Algorithm*, Vol. 3(2014), 549-553 (special issue, February 2014).

[4] P. Keevash, P.-S. Loh and B. Sudakov, Bounding the number of edges in permutation graphs, *Electronic J. Comb.*, **13**(2006), Research Paper 44.

[5] M. J.Atallah, G. K. Manacher and J. Urrutia, Finding a minimum independent dominating set in a permutation graph, *Discret Applied Mathematics*, **21**(3)(1988), 177-183.

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