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FURTHER RESULTS ON ZERO RING LABELING OF GRAPHS

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Abstract. Let $G = (V, E)$ be a graph with vertex set $V =: V(G)$, edge set $E =: E(G)$ and R^0 be a finite zero ring. An injective function $f : V(G) \to R^0$ is called a *zero ring labeling* of G if for every edge $uv \in E$ one has $f(u)+f(v) \neq 0$, where '0' is the additive identity of R^0. In this paper, we determine the optimal zero ring index for cycle graph and Petersen graph. In addition, we determine the necessary and sufficient condition for a finite graph of order 'n' to attain optimal zero ring index equal to 'n'.

1. Introduction

For all terminology and notation in graph theory and abstract algebra, not specifically defined in this paper, we refer the reader to the text-books by Harary [**4**] and Jacobson [**5**] respectively. Unless mentioned otherwise, all graphs considered in this paper are simple, connected and finite.

A ring in which the product of any two elements is 0, is called a *zero ring* and denoted by R^0, mathematically, $a \cdot b = 0, \ \forall \ a, b \in R^0$, where '0' is the additive identity of *zero ring*. One of the standard examples of *zero ring* is the set of 2×2 matrices defined as

$$\left\{ \begin{bmatrix} r & -r \\ r & -r \end{bmatrix} \right\},$$

where $r \in R$ and R is a commutative ring. Throughout the paper we shall denote the above set by $M_2^0(R)$. The above example implies the validity of the following remark.

Remark 1.1. For every positive integer n, there exists a zero ring of order n.

For more study on zero rings the reader is referred to [**6, 8, 9**].

Proposition 1.1. ([**9**]) *If* $n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots p_k^{a_k}$, *then the number of zero rings of order* n *are* $p(a_1) \cdot p(a_2) \cdots p(a_k)$, *where* $p(a_i)$ *denotes the partition of a natural number.*

Before going to the main section, first we recall some existing definitions and results to make the paper self contained. The following is the formal definition of *zero ring graph* which was introduced in [**7**].

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Definition 1.1. Let $(R^0, +, \cdot)$ be a finite zero ring, the zero ring graph, denoted by ΓR^0 is a graph whose vertices are the elements of zero ring R^0 and two distinct vertices x and y are adjacent if and only if $x + y \neq 0$, where '0' is the additive identity of R^0.

For example the zero ring graphs $\Gamma(M_2^0(\mathbb{Z}_4))$ and $\Gamma(M_2^0(\mathbb{Z}_5))$ are isomorphic to $K_4 - \{e\}$ and $C_4 + K_1$ respectively.

In this paper, we focus on vertex labeling of graphs by the elements of finite zero rings. One of the objectives of the paper is to label the vertices of those graphs which are not zero ring graphs as for instance, it is known that no cycle C_n of length $n \geq 4$ is a zero ring graph (cf.: [**7**]). Also, we determine the optimal zero ring index for cycle and Petersen graph. Another reason to study the optimal zero ring index is that whether an arbitrary graph of order n can be embedded in a zero ring graph of order n or not, towards approaching this we establish the necessary and sufficient condition for a finite graph of order 'n' to attain optimal zero ring index equal to 'n'.

The notion of zero ring labeling was introduced by Acharya et al. [**2**] and definition is as follows:

Definition 1.2. Let $G = (V, E)$ be a graph with vertex set $V =: V(G)$ and edge set $E =: E(G)$, and R^0 be a finite zero ring. An injective function $f : V(G) \to R^0$ is called a zero ring labeling of G if $f(u) + f(v) \neq 0$ for every edge $(uv) \in E$, where '0' is the additive identity of R^0.

In the contrast of above definition, the following results were proved in [**2**] and useful for this work.

Remark 1.2. If f is a zero ring labeling of a graph G, then f is a zero ring labeling of every subgraph of G.

Theorem 1.1. *Every finite graph admits a zero ring labeling.*

Theorem 1.2. *$\xi(K_n) = n$ if and only if $n = 2^{k_0}$, $k_0 > 1$.*

The following notion mainly stems from Theorem 1.1.

Definition 1.3. Let $G = (V, E)$ be a finite graph of order n. The *zero ring index* of G, denoted by $\xi(G)$, is the least positive integer n_0 such that there exists a zero ring R^0 of order n_0, for which G admits a zero ring labeling. Any zero ring labeling f of G is *optimal* if it uses a zero ring consisting of $\xi(G)$ elements.

In view of the above definition along with Theorem 1.1, it implies that

$$n \leq \xi(G) \leq 2^k, \tag{1.1}$$

where n is the order of graph and k is the ceiling of $\log_2 n$.

The bounds in Inequality (1.1) indicates the following problem of fundamental importance.

Problem 1.1. Characterize the graphs for which the bounds in Inequality (1.1) are attained. Also, determine all the zero rings which provide an optimal zero ring labeling for these graphs.

2. Main results

In this section, first, we study zero ring labeling for cycle and Petersen graphs which will help us to answer Problem 1.1. Further, we establish the necessary and sufficient condition for graphs to attain the lower bound in Inequality (1.1).

Theorem 2.1. *For each positive integer $n > 3$, $\xi(C_n) = n$.*

PROOF. Let us consider the cycle $C_n,\ n > 3$, whose vertices are labeled by $v_0, v_1, v_2, \ldots, v_{n-1}$ in a usual manner. Take the zero ring of order n, viz., $M_2^0(\mathbb{Z}_n)$ having the elements

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \ldots, \begin{bmatrix} n-1 & -(n-1) \\ n-1 & -(n-1) \end{bmatrix} \right\}.$$

To prove the desired result we need to tackle two cases for n, viz., n is even and n is odd.

Let n be even: Define $f : V(C_n) \to M_2^0(\mathbb{Z}_n)$ such that

$$f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix},\ \forall\ 0 \le i \le n-1.$$

One can easily verify that f is injective. We shall now show that f is a zero ring labeling.

Clearly for all $(v_i, v_{i+1}) \in E(C_n)$,

$$f(v_i) + f(v_{i+1}) = \begin{bmatrix} i+(i+1) & -(i+(i+1)) \\ i+(i+1) & -(i+(i+1)) \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\text{mod } n),$$

for all $0 \le i \le n-1$. Thus, f so defined is a zero ring labeling of C_n with respect to zero ring $M_2^0(\mathbb{Z}_n)$. In view of Definition 1.3, we conclude that $\xi(C_n) = n$.

Next, let n be odd: Define $f : V(C_n) \to M_2^0(\mathbb{Z}_n)$ such that

$$f(v_i) = \begin{cases} \begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, & 0 \le i \le \frac{n-1}{2} \\ \begin{bmatrix} i+1 & -(i+1) \\ i+1 & -(i+1) \end{bmatrix}, & \frac{n-1}{2} < i \le n-2 \\ \begin{bmatrix} \frac{i+2}{2} & -(\frac{i+2}{2}) \\ \frac{i+2}{2} & -(\frac{i+2}{2}) \end{bmatrix}, & i = n-1 \end{cases}$$

Clearly, f is an injective function and the sum of labels of end vertices for an edge is nonzero, i.e.,

$$f(v_i) + f(v_{i+1}) \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\text{mod n}),\ \forall\ i.$$

Again by Definition 1.3, we conclude that $\xi(C_n) = n$.

Consequently, f is a zero ring labeling with optimal zero ring index n. □

It should be noted here that, the above theorem indicates that lower bound in Inequality (1.1) is attained in case of cycle $C_n,\ n > 3$.

PROPOSITION 2.1. *For the Petersen graph* $P(5,2)$,

$$\xi(P(5,2)) = 10.$$

PROOF. Consider the Petersen graph $P(5,2)$. To prove the desired result, we shall use the fact that the complement of $P(5,2)$ is line graph of K_5, and hence contains a 4-matching. Therefore, $P(5,2)$ is a subgraph of $\Gamma M_2^0(\mathbb{Z}_{10})$. □

From the above analysis, it is clear that the sharp bounds in Inequality (1.1) are attained. Now in order to show that $\xi(G)$ is neither equal to 'n' nor 2^k, we have the following example.

EXAMPLE 2.1. Let us take $G \cong K_5 - \{e\}$, the zero ring labeling of G can be obtained using the zero ring $M_2^0(\mathbb{Z}_6)$ and clearly the optimal zero ring index is 6, which shows that $5 < \xi(G) < 8$.

In the next theorem we establish the sufficient condition for an arbitrary graph to attain the lower bound in Inequality (1.1).

Theorem 2.2. *Let G be a graph of order n and if $n = 2^{k_0}$, $k_0 \geq 1$, then $\xi(G) = n$.*

Proof. The result follows directly from the Theorem 1.2 along with the fact that every graph is a subgraph of complete graph. $\square$

Theorem 2.3. *For a graph G of order n, $\xi(G) = n$ if and only if G is a spanning subgraph of ΓR^0.*

Proof. Necessity: Assume that $\xi(G) = n$ with respect to labeling f. Our aim is to show that G is a spanning subgraph of zero ring graph ΓR^0 of order n. On the contrary, suppose that G is not a spanning subgraph of ΓR^0, then the only possibility is that G is not a subgraph of ΓR^0, as $|V(G)| = |V(\Gamma R^0)|$. Without loss of generality, let us assume that G is obtained by adding at least one edge e in ΓR^0, but ΓR^0 is a maximal in sense that for all $(v_i, v_j) \in E(\Gamma R^0)$, $v_i + v_j \neq 0$, where 0 is the additive identity of R^0. Therefore, so obtained G (by adding at least one edge in ΓR^0) consists of at least one pair of vertices, for which $f(v_i) + f(v_j) = 0$, this implies that the optimal zero ring index can not be n, i.e., $\xi(G) > n$, which is a contradiction. Hence G must be a spanning subgraph of a zero ring graph.

Sufficiency: Assume that G is a spanning subgraph of zero ring graph then using Remark 1.2, one can easily verify that any injective function yields an optimal zero ring labeling with optimal zero ring index $\xi(G) = n$. Thus the proof follows. $\square$

Next theorem establishes the relation between zero ring index of a graph and its spanning subgraph.

Theorem 2.4. *Let G be a graph of order n and ΓR^0 be a zero ring graph of order n. Then for a spanning subgraph H of G, $\xi(H) \leq \xi(G)$.*

Proof. Let G be a graph of order n and f be a zero ring labeling of G. In view of Remark 1.2, it is clear that f is a zero ring labeling of H as well. Now, if G is a spanning subgraph of a zero ring graph ΓR^0, then $\xi(G) = n$, consequently $\xi(H) = n$. On the other hand, if G is not a spanning subgraph of ΓR^0, then $\xi(G) > n$, however H may or may not be a spanning subgraph of ΓR^0, then $\xi(H) \leq \xi(G)$. Hence the result. $\square$

The following example illustrates the Theorem 2.4.

Example 2.2. Let us take $G \cong K_7$ and let $H \cong C_7$. Note that K_7 is neither isomorphic to $\Gamma(M_2^0(\mathbb{Z}_7)) \cong K_7 - \{3e\}$ nor a spanning subgraph of $\Gamma(M_2^0(\mathbb{Z}_7))$. However, C_7 is a spanning subgraph of $\Gamma(M_2^0(\mathbb{Z}_7))$. Clearly, $\xi(K_7) = 8$ and $\xi(C_7) = 7$, this implies that $\xi(H) < \xi(G)$.
On the other hand, if $G \cong K_8$ and $H \cong C_8$, then both G and H are spanning subgraph of $\Gamma(M_2^0(\mathbb{Z}_2[x]/\langle x^3\rangle))$. Now it is easy to see that $\xi(K_8) = 8$ and $\xi(C_8) = 8$, this implies that $\xi(H) = \xi(G)$.

The *clique* of a graph G is the complete subgraph of G and the clique number $\omega(G)$ is the size of maximum clique in a graph.

Theorem 2.5. *Let G be a graph of odd order n, where n is either the product of distinct primes or $n = p^2$. Then $\xi(G) = n$ if and only if $\omega(G) \leq \frac{n+1}{2}$.*

Proof. Necessity: Assume that $\xi(G) = n$ and our aim is to show that $\omega(G) \leq \frac{n+1}{2}$. Suppose on contrary that $\omega(G) > \frac{n+1}{2}$ and let
$(v_0, v_1, v_2, v_3, \ldots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}})$ be a clique of size $> \frac{n+1}{2}$ in G. Firstly, when n is the product of distinct primes there exists only one zero ring of order n, viz., $M_2^0(\mathbb{Z}_n)$ having the elements

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}, \ldots, \begin{bmatrix} n-1 & -(n-1) \\ n-1 & -(n-1) \end{bmatrix} \right\}.$$

Define $f : V(G) \to M_2^0(\mathbb{Z}_n)$ such that

$$f(v_i) = \begin{bmatrix} i & -i \\ i & -i \end{bmatrix},\ 0 \leq i \leq n-1$$

clearly, f is injective and the vertices $v_0, v_1, \ldots, v_{\frac{n-1}{2}}, v_{\frac{n+1}{2}}$ of clique are labeled by the elements

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \ldots, \begin{bmatrix} \frac{n-1}{2} & -(\frac{n-1}{2}) \\ \frac{n-1}{2} & -(\frac{n-1}{2}) \end{bmatrix}, \begin{bmatrix} \frac{n+1}{2} & -(\frac{n+1}{2}) \\ \frac{n+1}{2} & -(\frac{n+1}{2}) \end{bmatrix}$$

respectively. However, the pair

$$\begin{bmatrix} \frac{n-1}{2} & -(\frac{n-1}{2}) \\ \frac{n-1}{2} & -(\frac{n-1}{2}) \end{bmatrix} \text{and} \begin{bmatrix} \frac{n+1}{2} & -(\frac{n+1}{2}) \\ \frac{n+1}{2} & -(\frac{n+1}{2}) \end{bmatrix}$$

produces a zero sum modulo n. Similarly, if we choose any other injective function then we select a subset of zero ring of cardinality $> \frac{n+1}{2}$ to label the vertices of clique in which there always exists at least one such pair which produces the zero sum modulo n. Consequently, we are unable to give an assignment to the clique of size $> \frac{n+1}{2}$ with respect to $M_2^0(\mathbb{Z}_n)$. Thus f can not be a zero ring labeling with optimal zero ring index n, which is a contradiction. Hence $\omega(G) \leq \frac{n+1}{2}$.

Secondly, when $n = p^2$. Due to assumption $\xi(G) = n$ and using Theorem 2.3, it follows that G is a spanning subgraph of zero ring graph ΓR^0. We know that if H is a spanning subgraph of a graph G', then $\omega(H) \leq \omega(G')$. Therefore, the relation $\omega(G) \leq \omega(\Gamma R^0)$ holds for G and ΓR^0. Now, in order to show the result, it suffices to find the clique of $\Gamma(R^0)$. Now, in view of Proposition 1.1, there exist exactly two zero rings of order p^2, viz., $M_2^0(\mathbb{Z}_{p^2})$ and $M_2^0(\mathbb{Z}_p[x]/\langle x^2\rangle)$. Due to [**7**, Theorem 2.28], $\Gamma(M_2^0(\mathbb{Z}_{p^2})) \cong \Gamma(M_2^0(\mathbb{Z}_p[x]/\langle x^2\rangle))$. Therefore, both have the same clique number, that is $\omega(\Gamma(M_2^0(\mathbb{Z}_{p^2})) = \omega(\Gamma(M_2^0(\mathbb{Z}_p[x]/\langle x^2\rangle)) = \frac{p^2+1}{2}$. Hence, $\omega(\Gamma R^0) = \frac{p^2+1}{2}$ for both the zero ring graphs. The desired result follows due to the relation $\omega(G) \leq \omega(\Gamma R^0)$.

Sufficiency: Assume that $\omega(G) \leq \frac{n+1}{2}$ and our aim is to show that $\xi(G) = n$. Let, if possible $\xi(G) \neq n$, i.e., $\xi(G) > n$. Firstly, when n is the product of distinct primes, there is only one zero ring, viz., $M_2^0(\mathbb{Z}_n)$. Due to Theorem 2.3, $\xi(G) > n$ implies that G can not be a spanning subgraph of $\Gamma(M_2^0(\mathbb{Z}_n))$. Since $|V(G)| = n = |V\Gamma(M_2^0(\mathbb{Z}_n))|$, the only possibility for G is that it is not a subgraph of $\Gamma(M_2^0(\mathbb{Z}_n))$. Without loss of generality, let G be obtained from $\Gamma(M_2^0(\mathbb{Z}_n))$ by adding at least one edge. By [**7**] we know that $\Gamma(M_2^0(\mathbb{Z}_n))$ always have an induced sub graph $H \cong K_{\frac{n+3}{2}} \setminus \{e\}$, (where, e is an edge not appearing in $\Gamma(M_2^0(\mathbb{Z}_n))$). Thus, so obtained G consists of the clique of size $\geq \frac{n+3}{2}$, that is $\omega(G) > \frac{n+1}{2}$, which is a contradiction to our assumption. Therefore, $\xi(G) = n$.

Secondly, when $n = p^2$, here also we shall prove the result by contradiction. Suppose on contrary that $\xi(G) > n$, by following the procedure as done in the previous part, we get that G can not be a spanning subgraph of $\Gamma(M_2^0(\mathbb{Z}_{p^2}))$ and $\Gamma(M_2^0(\mathbb{Z}_p[x]/\langle x^2\rangle))$ (as $n = p^2$). Although, due to [**7**, Theorem 2.28], $\Gamma(M_2^0(\mathbb{Z}_{p^2})) \cong \Gamma(M_2^0(\mathbb{Z}_p[x]/\langle x^2\rangle))$. Thus, G can not be a spanning sub graph of $\Gamma(M_2^0(\mathbb{Z}_{p^2}))$. Now it is not difficult to prove that G contains the clique of size $\geq \frac{p^2+3}{2}$ that is $\omega(G) > \frac{p^2+1}{2}$, which is a contradiction to our assumption. Therefore, $\xi(G) = n$. □

THEOREM 2.6. *Let G be a graph of even order n, where n is the product of distinct primes. Then $\xi(G) = n$ if and only if $\omega(G) \leq \frac{n}{2} + 1$.*

PROOF. The proof of the result can be given by the arguments analogous to those used in Theorem 2.5. □

It should be noted that if $n = 2^2$, then $\xi(G) = 4$ which does not imply that $\omega(G) \leq \frac{n}{2} + 1$, as for instance; Consider K_4. Then $\xi(K_4) = 4 = \omega(K_4)$ which is not less than or equal to 3.

Invoking the Theorems 2.5 and 2.6 we have the following result.

THEOREM 2.7. *Let G be a graph of order n, where n is either the product of distinct primes or $n = p^2 \neq 4$. Then $\xi(G) = n$ if and only if $\omega(G) \leq \lceil \frac{n}{2} + 1 \rceil$.*

We conclude the paper with following remark:

REMARK 2.1. For every positive integer n there exists a graph of order n with $\xi(G) = n$.

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