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GRADINGS ON SEMIDIHEDRAL BLOCKS WITH THREE SIMPLE MODULES

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Abstract. In this paper we show that tame blocks of group algebras with semidihedral defect groups and three isomorphism classes of simple modules can be non-trivially graded. We prove this by using the transfer of gradings via derived equivalences.

1. Introduction and preliminaries

Let A be an algebra over a field k. We say that A is a graded algebra if A is the direct sum of subspaces $A = \bigoplus_{i\in\mathbb{Z}} A_i$, such that $A_iA_j \subset A_{i+j}$, $i, j \in \mathbb{Z}$. The subspace A_i is said to be the homogeneous subspace of degree i. It is obvious that we can always trivially grade A by setting $A_0 = A$. In this paper we study the problem of existence of non-trivial gradings on blocks of group algebras with semidihedral defect groups and three simple modules. We refer the reader to [1] for details on defect groups of blocks of group algebras. Tame blocks of group algebras appear only for group algebras over fields of characteristic 2, so for the remainder of this paper we will assume that the field we work over is of characteristic 2. All algebras in this paper are finite dimensional algebras over the field k, and all modules will be left modules. The category of finite dimensional A–modules is denoted by A–mod and the full subcategory of finite dimensional projective A–modules is denoted by P_A. The derived category of bounded complexes over A–mod is

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denoted by $D^b(A)$, and the homotopy category of bounded complexes over P_A will be denoted by $K^b(P_A)$.

For some algebras, such as group algebras, it is not obvious how one can construct non-trivial gradings on these algebras. More complex methods, such as transfer of gradings via derived and stable equivalences ([10, 5, 6]), had to be developed to introduce non-trivial gradings on certain blocks of group algebras. The aim of this paper is to show how one can use transfer of gradings via derived equivalences to grade tame blocks with semidihedral defect groups and three simple modules.

Let A and B be two symmetric algebras over a field k and let us assume that A is a graded algebra. The following theorem is due to Rouquier.

Theorem 1.1 ([10, Theorem 6.3]). *Let A and B be as above. Let T be a tilting complex of A-modules that induces a derived equivalence between A and B. Then there exists a grading on B and a structure of a graded complex T' on T, such that T' induces an equivalence between the derived categories of graded A-modules and graded B-modules.*

This theorem tells us that derived equivalences are compatible with gradings, that is, gradings can be transferred between symmetric algebras via derived equivalences.

For a given tilting complex T of A-modules, which is a bounded complex of finitely generated projective A-modules, there exists a structure of a complex of graded A-modules T' on T. If T is a tilting complex that tilts from A to B, then $\mathrm{End}_{K^b(P_A)}(T) \cong B^{op}$. Viewing T as a graded complex T', and by computing its endomorphism ring as a graded object, we get a graded algebra which is isomorphic to the opposite algebra of the algebra B. We notice here that the choice of a grading on T' is unique up to shifting the grading of each indecomposable summand of T'. This follows from the fact that if we have two different gradings on an indecomposable module (bounded complex), then they differ only by a shift (see [3, Lemma 2.5.3]). We refer the reader to [9, 5, 6] for details on categories of graded modules and their derived categories, and [11] for basics of homological algebra.

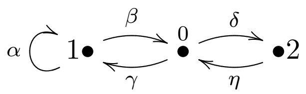
1.1. **Semidihedral blocks.** Any block with a semidihedral defect group and three isomorphism classes of simple modules is Morita equivalent to some algebra from the following list (cf. [7]).

(1) For any $r \geqslant 1$ let $SD(3A)_1^r$ be the algebra defined by the quiver and relations

$$\begin{tikzcd} 1 \bullet \arrow[r, bend left, "\beta"] & \overset{0}{\bullet} \arrow[l, bend left, "\gamma"] \arrow[r, bend left, "\delta"] & \bullet 2 \arrow[l, bend left, "\eta"] \end{tikzcd}$$

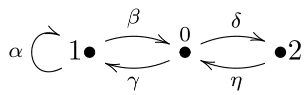
$$\begin{aligned} \beta\gamma &= 0, \\ \delta\eta\delta &= (\gamma\beta\delta\eta)^{r-1}\gamma\beta\delta, \\ \eta\delta\eta &= (\eta\gamma\beta\delta)^{r-1}\eta\gamma\beta. \end{aligned}$$

(2) For any $r \geqslant 2$ let $SD(3B)_1^r$ be the algebra defined by the quiver and relations



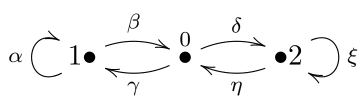
$$\alpha\beta = \gamma\alpha = \beta\gamma = 0,$$
$$\alpha^r = \beta\delta\eta\gamma,$$
$$\eta\delta\eta = \eta\gamma\beta,\ \delta\eta\delta = \gamma\beta\delta.$$

(3) For any $r \geqslant 3$ let $SD(3B)_2^r$ be the algebra defined by the quiver and relations



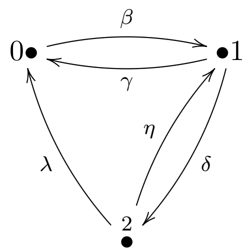
$$\eta\delta = 0,\ \beta\gamma = \alpha^{r-1},$$
$$\gamma\alpha = \delta\eta\gamma\beta\delta\eta\gamma,$$
$$\alpha\beta = \beta\delta\eta\gamma\beta\delta\eta.$$

(4) For any $r \geqslant 3$ let $SD(3D)^r$ be the algebra defined by the quiver and relations



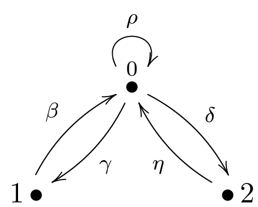
$$\delta\xi = \xi\eta = \eta\delta = 0,$$
$$\beta\gamma = \alpha^{r-1},\ \gamma\alpha = \delta\eta\gamma,$$
$$\alpha\beta = \beta\delta\eta,\ \xi^2 = \eta\gamma\beta\delta.$$

(5) For any integer $r \geqslant 2$ let $SD(3H)^r$ be defined by the quiver and relations



$$\delta\lambda = \gamma\beta\gamma,\ \lambda\beta = (\eta\delta)^{r-1}\eta,$$
$$\beta\delta\eta = 0,\ \gamma\beta\delta = 0,\ \eta\gamma = 0.$$

(6) For any integers $r, s \geqslant 2$ consider the quiver and relations



$$\beta\rho = \rho\delta = \eta\rho = \rho\gamma = 0,$$
$$\gamma\beta = \delta\eta,\ (\gamma\beta)^r = \rho^s,$$
$$(\beta\gamma)^{r-1}\beta\delta = 0,\ (\eta\delta)^{r-1}\eta\gamma = 0.$$

Because blocks of semidihedral type with defect n can only occur in this family for parameters $\{r, s\} = \{2, 2^{n-2}\}$ (cf. [7]) we have to distinguish two cases.

(a) For any $s \geqslant 2$ let $SD(3C)^s_{2,I}$ be the above algebra when $r = 2$.
(b) For any $r \geqslant 2$ let $SD(3C)^r_{2,II}$ be the above algebra when $s = 2$.

We recommend [2] and [4] as a good introduction to path algebras of quivers.

Holm proved in [8] that all blocks with common semidihedral defect group and three isomorphism classes of simple modules are derived equivalent. We will use the tilting complexes given in [8] to transfer gradings via derived equivalences.

The rest of the paper is devoted to proving the following theorem.

Theorem 1.2. *Let A be a tame block of group algebras with semidihedral defect groups and three isomorphism classes of simple modules. There exists a non-trivial grading on A.*

2. Transfer of gradings via derived equivalences

For the remainder of this paper, if we say that an algebra given by a quiver and relations is graded, we will assume that it is graded in such a way that the arrows and the vertices of its quiver are homogeneous. Also, right multiplication by a given path ρ will be denoted by the same letter ρ.

Let us assume that $SD(3A)^r_1$ is graded and that $\deg(\beta) = d_2$, $\deg(\gamma) = d_3$, $\deg(\delta) = d_4$ and $\deg(\eta) = d_5$. We write Σ for $d_2 + d_3 + d_4 + d_5$.

The graded radical layers of the projective indecomposable $SD(3A)^r_1$-modules are as follows (the numbers to the left and right denote the degrees of the corresponding layers):

$$
\begin{array}{ccccc}
 & & S_0 & & \\
d_2 & S_1 & & S_2 & d_5 \\
d_2 + d_3 & S_0 & & S_0 & d_5 + d_4 \\
d_2 + d_3 + d_5 & S_2 & & S_1 & d_5 + d_4 + d_2 \\
\Sigma & S_0 & & S_0 & \Sigma \\
 & \vdots & & \vdots & \\
(r-1)\Sigma + d_2 & S_1 & & S_2 & (r-1)\Sigma + d_5 \\
(r-1)\Sigma + d_2 + d_3 & S_0 & & S_0 & (r-1)\Sigma + d_5 + d_4 \\
(r-1)\Sigma + d_2 + d_3 + d_5 & S_2 & & S_1 & (r-1)\Sigma + d_5 + d_4 + d_2, \\
2d_4 + 2d_5 & & S_0 & &
\end{array}
,
$$

$$
\begin{array}{cc}
 & S_1 \\
d_3 & S_0 \\
d_3+d_5 & S_2 \\
d_3+d_5+d_4 & S_0 \\
\Sigma & S_1 \\
 & \vdots \\
(r-1)\Sigma+d_3 & S_0 \\
(r-1)\Sigma+d_3+d_5 & S_2 \\
r\Sigma-d_2 & S_0 \\
2d_4+2d_5 & S_1
\end{array}
,\qquad
\begin{array}{ccccc}
 & & S_2 & & \\
 & & S_0 & & d_4 \\
 & & & S_1 & d_4+d_2 \\
 & & & S_0 & d_4+d_2+d_3 \\
 & & & S_2 & \Sigma \\
d_4+d_5 & S_2 & & \vdots & \\
 & & & S_0 & (r-1)\Sigma+d_4 \\
 & & & S_1 & (r-1)\Sigma+d_4+d_2 \\
 & & S_0 & & r\Sigma-d_5 \\
 & & S_2 & & 2d_4+2d_5
\end{array}
.
$$

Since the relations of $SD(3A)_1^r$ are homogeneous, the following equation holds

$$2d_4+2d_5=r\Sigma.$$

A tilting complex that tilts from $SD(3A)_1^r$ to $SD(3B)_1^r$ is given by $T := T_0 \oplus T_1 \oplus T_2$, where T_1 is the stalk complex with P_1 in degree 0, T_2 is given by

$$0 \longrightarrow P_1 \xrightarrow{\beta\delta} P_2\langle d_2+d_4\rangle \longrightarrow 0\ ,$$

with non-zero terms in degrees 0 and 1, and T_0 is given by

$$0 \longrightarrow P_1\langle -d_2\rangle \oplus P_1\langle -d_2-d_4-d_5\rangle \xrightarrow{(\beta,\beta\delta\eta)} P_0 \longrightarrow 0\ ,$$

with non-zero terms in degrees 0 and 1.

It is obvious that $\mathrm{Endgr}_{K^b(P_{SD(3A)})}(T_1) \cong \bigoplus_{t=0}^{r} k\langle -t\Sigma\rangle$. Therefore, $\deg(\alpha) = \Sigma$ in the quiver of $SD(3B)_1^r$. Also, $\mathrm{Homgr}_{K^b(P_{SD(3A)})}(T_1,T_2) \cong k\langle -2d_4+2d_5\rangle$ implies that $\deg(\beta)+\deg(\delta) = 2d_4+2d_5 = r\Sigma$. Similarly, $\mathrm{Homgr}_{K^b(P_{SD(3A)})}(T_1,T_0) \cong k\langle -(r\Sigma+d_2)\rangle \oplus k\langle -(r\Sigma+d_2+d_4+d_5)\rangle$. It follows that $\{\deg(\beta), \deg(\beta\delta\eta)\} = \{r\Sigma+d_2, r\Sigma+d_2+d_4+d_5\}$.

We have that $\mathrm{Homgr}_{K^b(P_{SD(3A)})}(T_0,T_1) \cong k\langle d_2\rangle \oplus k\langle d_2+d_4+d_5\rangle)$, because the maps that are not homotopic to zero have to map $P_1\langle -d_2\rangle$ or $P_1\langle -d_2-d_4-d_5\rangle$ onto P_1. We have that $\{\deg(\gamma), \deg(\delta\eta\gamma)\} = \{-d_2, -d_2-d_4-d_5\}$.

If $\deg(\beta) = r\Sigma+d_2$, it follows that $\deg(\delta) = -d_2$, $\deg(\eta) = d_2+d_4+d_5$ and $\deg(\gamma) = -d_2-d_4-d_5$.

Therefore, the graded quiver of $SD(3B)_1^r$ is given by

$$
\begin{tikzcd}
1 \bullet \arrow[loop left, "\Sigma"] \arrow[r, bend left, "d_2+r\Sigma"] & \overset{0}{\bullet} \arrow[l, bend left, "-d_2-d_4-d_5"] \arrow[r, bend left, "-d_2"] & \bullet 2 \arrow[l, bend left, "d_2+d_4+d_5"]
\end{tikzcd}
$$

If $\deg(\beta) = r\Sigma + d_2 + d_4 + d_5$, then $\deg(\eta) = d_2$, $\deg(\delta) = -d_2 - d_4 - d_5$ and $\deg(\gamma) = -d_2$. This does not give us a grading on $SD(3B)_1^r$ because the relations are not homogeneous.

We proceed by transferring gradings from $SD(3A)_1^r$ to $SD(3B)_2^r$. A tilting complex that tilts from $SD(3A)_1^r$ to $SD(3B)_2^r$ is given by $T := T_0 \oplus T_1 \oplus T_2$, where T_1 is the stalk complex with P_2 in degree 0, T_2 is given by

$$0 \longrightarrow P_2 \xrightarrow{\eta\gamma} P_1\langle d_3 + d_5\rangle \longrightarrow 0 \ ,$$

with non-zero terms in degrees 0 and 1, and T_0 is given by

$$0 \longrightarrow P_2\langle -d_5\rangle \oplus P_2\langle -d_2 - d_3 - d_5\rangle \xrightarrow{(\eta,\eta\gamma\beta)} P_0 \longrightarrow 0 \ ,$$

with non-zero terms in degrees 0 and 1.

Since $\mathrm{Endgr}_{K^b(P_{SD(3A)})}(T_1) \cong \bigoplus_{t=0}^{r} k\langle -t\Sigma\rangle \oplus k\langle -(d_4 + d_5)\rangle$, we conclude that $\deg(\alpha) = \Sigma$ and $\deg(\beta)+\deg(\delta)+\deg(\eta)+\deg(\gamma) = d_4+d_5$ in the quiver of $SD(3B)_2^r$. From these equalities and the relations of $SD(3B)_2^r$ we get that in the quiver of $SD(3B)_2^r$, $\deg(\delta) + \deg(\eta) = d_2 + d_3$ and $\deg(\beta) + \deg(\gamma) = d_4 + d_5 - d_2 - d_3$.

Also, from $\mathrm{Homgr}_{K^b(P_{SD(3A)})}(T_2, T_1) \cong k\langle 0\rangle \oplus k\langle -(d_4 + d_5)\rangle$ it follows easily that $\{\deg(\eta\gamma), \deg(\eta\gamma\beta\delta\eta\gamma)\} = \{0, d_4 + d_5\}$. Combining this and the results of the previous paragraph we get that $\deg(\eta) + \deg(\gamma) = 0$ and $\deg(\beta) + \deg(\delta) = d_4 + d_5$.

The simple module S_2 appears four times as a composition factor of $\ker(\eta, \eta\gamma\beta)$. Thus, the graded vector space $\mathrm{Homgr}_{K^b(P_{SD(3A)})}(T_1, T_0)$ is isomorphic to the sum $k\langle d_4+2d_5\rangle \oplus k\langle d_4+2d_5+d_2+d_3\rangle \oplus k\langle 2d_4+3d_5\rangle \oplus k\langle 2d_4+3d_5+d_2+d_3\rangle$. We conclude that $\deg(\beta) = d_4+2d_5$, $\deg(\gamma) = -d_2-d_3-d_5$, $\deg(\delta) = -d_5$ and $\deg(\eta) = d_2 + d_3 + d_5$.

Therefore, the resulting graded quiver of $SD(3B)_2^r$ is given by

$$\begin{tikzcd} 1 \bullet \arrow[loop left, "\Sigma"] \arrow[r, bend left, "d_4+2d_5"] & \overset{0}{\bullet} \arrow[l, bend left, "-d_2-d_3-d_5"] \arrow[r, bend left, "-d_5"] & \bullet 2 \arrow[l, bend left, "d_2+d_3+d_5"] \end{tikzcd}$$

We now proceed by transferring gradings from $SD(3B)_2^r$ to $SD(3D)^r$. Let us assume that $SD(3B)_2^r$ is graded in such a way that $\deg(\alpha) = d_1$, $\deg(\beta) = d_2$, $\deg(\gamma) = d_3$, $\deg(\delta) = d_4$ and $\deg(\eta) = d_5$ in the quiver of $SD(3B)_2^r$. We write Σ for $d_2 + d_3 + d_4 + d_5$. Since the relations of $SD(3B)_2^r$ are homogeneous, we have that $2\Sigma = d_1 + d_2 + d_3$, and $(r - 1)d_1 = d_2 + d_3$.

The graded radical layers of the projective indecomposable $SD(3B)_2^r$-modules are:

$$
\begin{array}{ccccc}
 & & S_0 & & \\
d_2 & S_1 & & S_2 & d_5 \\
d_2+d_3 & S_0 & & S_0 & d_4+d_5 \\
d_2+d_3+d_5 & S_2 & & S_1 & d_2+d_4+d_5 \\
\Sigma & S_0 & & S_0 & \Sigma \\
d_2+\Sigma & S_1 & & S_2 & d_5+\Sigma \\
d_2+d_3+\Sigma & S_0 & & S_0 & d_4+d_5+\Sigma \\
d_2+d_3+d_5+\Sigma & S_2 & & S_1 & d_2+d_4+d_5+\Sigma \\
2\Sigma & & S_0 & &
\end{array} ,
$$

$$
\begin{array}{ccccc}
 & & S_1 & & \\
d_1 & S_1 & & S_0 & d_3 \\
2d_1 & S_1 & & S_2 & d_3+d_5 \\
3d_1 & S_1 & & S_0 & d_3+d_4+d_5 \\
 & \vdots & & S_1 & \Sigma \\
(r-3)d_1 & S_1 & & S_0 & \Sigma+d_3 \\
(r-2)d_1 & S_1 & & S_2 & \Sigma+d_3+d_5 \\
d_2+d_3 & S_1 & & S_0 & d_1+d_3 \\
2\Sigma & & S_1 & &
\end{array} ,
\qquad
\begin{array}{cc}
S_2 & \\
S_0 & d_4 \\
S_1 & d_2+d_4 \\
S_0 & d_2+d_3+d_4 \\
S_2 & \Sigma \\
S_0 & \Sigma+d_4 \\
S_1 & \Sigma+d_4+d_2 \\
S_0 & \Sigma+d_4+d_2+d_3 \\
S_2 & 2\Sigma
\end{array} .
$$

A tilting complex that tilts from $SD(3B)_2^r$ to $SD(3D)^r$ is given by $T := T_0 \oplus T_1 \oplus T_2$, where T_2 is the stalk complex with P_2 in degree 0, T_1 is given by

$$0 \longrightarrow P_2 \xrightarrow{\eta\gamma} P_1\langle d_3+d_5\rangle \longrightarrow 0 \ ,$$

with non-zero terms in degrees 0 and 1, and T_0 is given by

$$0 \longrightarrow P_2\langle -d_5\rangle \oplus P_2\langle -d_2-d_3-d_5\rangle \xrightarrow{(\eta,\eta\gamma\beta)} P_0 \longrightarrow 0 \ ,$$

with non-zero terms in degrees 0 and 1.

Since $\mathrm{Endgr}_{K^b(P_{SD(3B)_2})}(T_2) \cong k\langle 0\rangle \oplus k\langle -\Sigma\rangle \oplus k\langle -2\Sigma\rangle$, we have that $\deg(\xi) = \Sigma$ in the quiver of $SD(3D)^r$.

Also, $\mathrm{Homgr}_{K^b(P_{SD(3B)_2})}(T_2, T_1) \cong \mathrm{Homgr}_{SD(3B)_2}(P_2, \ker(\eta\gamma))$ which is isomorphic to $k\langle -2\Sigma\rangle$. It follows that $\deg(\eta)+\deg(\gamma) = d_1+d_2+d_3$. Similarly, $\mathrm{Homgr}_{K^b(P_{SD(3B)_2})}(T_1, T_2) \cong k\langle 0\rangle$ implies that $\deg(\beta)+\deg(\delta) = 0$. The same argument will lead us to the conclusion that $\mathrm{Homgr}_{K^b(P_{SD(3B)_2})}(T_2, T_0) \cong k\langle -(d_5+2\Sigma)\rangle \oplus k\langle -(d_2+d_3+d_5+2\Sigma)\rangle$. This means that $\{\deg(\eta), \deg(\eta\gamma\beta)\} = \{d_5+2\Sigma, d_2+d_3+d_5+2\Sigma\}$.

Similar calculations give us that $\mathrm{Homgr}_{K^b(P_{SD(3B)_2})}(T_0, T_2) \cong k\langle d_5\rangle \oplus k\langle d_2+d_3+d_5\rangle$. This implies that $\{\deg(\delta), \deg(\gamma\beta\delta)\} = \{-d_5, -d_2-d_3-d_5\}$.

If $\deg(\eta) = d_5 + 2\Sigma$, it follows that $\deg(\beta) = d_2 + d_3 + d_5$, $\deg(\delta) = -d_2 - d_3 - d_5$, $\deg(\gamma) = -d_5$ and $\deg(\alpha) = d_1$. The resulting graded quiver of $SD(3D)^r$ is given by

$$\begin{tikzcd} 1 \arrow[loop left, "d_1"] \arrow[r, bend left, "d_2+d_3+d_5"] & \overset{0}{\bullet} \arrow[l, bend left, "-d_5"] \arrow[r, bend left, "-d_2-d_3-d_5"] & 2 \arrow[l, bend left, "d_5+2\Sigma"] \arrow[loop right, "\Sigma"] \end{tikzcd}$$

If $\deg(\eta) = d_2 + d_3 + d_5 + 2\Sigma$, then the relations are not homogeneous and we conclude that in this case we will not have a grading on $SD(3D)^r$.

To put a grading on the remaining three families we will start from $SD(3C)^r_{2,II}$, which can be graded easily, and transfer gradings to the other two families of algebras.

Let us assume that $SD(3C)^r_{2,II}$ is graded and that $\deg(\beta) = d_2$, $\deg(\gamma) = d_3$, $\deg(\delta) = d_4$, $\deg(\eta) = d_5$ and $\deg(\rho) = d_6$. We write Σ for $d_2+d_3+d_4+d_5$. The graded radical layers of the projective indecomposable $SD(3C)^r_{2,II}$-modules are:

$$\begin{array}{ccccc}
 & & S_1 & & \\
d_3 & & S_0 & & \\
d_2+d_3 & S_1 & & S_2 & d_5+d_3 \\
d_2+2d_3 & & S_0 & & \\
2d_2+2d_3 & S_1 & & S_2 & d_3+d_4+2d_5 \\
 & & S_0 & & 2d_2+3d_3 \\
 & & \vdots & & \\
(r-1)(d_2+d_3) & S_1 & & S_2 & (r-1)d_5+(r-2)d_4+d_3 \\
(r-1)d_2+rd_3 & & S_0 & & \\
r(d_2+d_3) & & S_1 & &
\end{array},$$

$$\begin{array}{ccccc}
 & & S_2 & & \\
d_4 & & S_0 & & \\
d_2+d_4 & S_1 & & S_2 & d_5+d_4 \\
d_5+2d_4 & & S_0 & & \\
d_2+2d_4+d_5 & S_1 & & S_2 & 2d_4+2d_5 \\
 & & S_0 & & 2d_5+3d_4 \\
 & & \vdots & & \\
(r-1)d_4+(r-2)d_5+d_2 & S_1 & & S_2 & (r-1)(d_4+d_5) \\
(r-1)d_5+rd_4 & & S_0 & & \\
r(d_2+d_3) & & S_2 & &
\end{array},$$

$$
\begin{array}{ccccccccc}
 & & S_0 & & & & & & \\
d_2 & S_1 & & S_2 & d_5 & & & & \\
d_2+d_3 & & S_0 & & & & & & \\
2d_2+d_3 & S_1 & & S_2 & d_4+2d_5 & \oplus & S_0 & d_6 & \\
2d_2+2d_3 & & S_0 & & & & & & . \\
 & & \vdots & & & & & & \\
(r-1)d_3+rd_2 & S_1 & & S_2 & (r-1)d_4+rd_5 & & & & \\
r(d_2+d_3) & & S_0 & & & & & &
\end{array}
$$

In the structure of P_0, the rightmost copy of S_0 is in the second radical layer, and at the same time, in the second socle layer. Hence, $\operatorname{rad} P_0/\operatorname{soc} P_0$ is a direct sum of two modules, one of which is S_0.

Since the relations of $SD(3C)^r_{2,II}$ are homogeneous the following equations hold

$$
\begin{aligned}
d_2+d_3 &= d_4+d_5,\\
r(d_2+d_3) &= 2d_6.
\end{aligned}
$$

A tilting complex that tilts from $SD(3C)^r_{2,II}$ to $SD(3H)^r$ is given by the direct sum $T := T_0 \oplus T_1 \oplus T_2$, where T_2 is the stalk complex with P_1 in degree 0, T_1 is the stalk complex with P_0 in degree 0, and T_0 is given by

$$0 \longrightarrow P_0 \xrightarrow{\delta} P_2\langle d_4\rangle \longrightarrow 0\ ,$$

with non-zero terms in degrees 0 and 1.

We easily compute that

$$\operatorname{Endgr}_{K^b(P_{SD(3C)^r_{2,II}})}(T_2) \cong \bigoplus_{t=1}^{r} k\langle -t(d_2+d_3)\rangle,$$

and that $\operatorname{Endgr}_{K^b(P_{SD(3C)^r_{2,II}})}(T_1) \cong \bigoplus_{t=1}^{r} k\langle -t(d_2+d_3)\rangle \oplus k\langle -d_6\rangle$. From this we deduce that $\deg(\eta)+\deg(\delta) = d_2+d_3$ and $\deg(\gamma)+\deg(\beta) = d_6$ in the quiver of $SD(3H)^r$.

Also, from $\operatorname{Homgr}_{K^b(P_{SD(3C)^r_{2,II}})}(T_1,T_2) \cong \bigoplus_{t=0}^{r-1} k\langle -(d_3+t(d_2+d_3))\rangle$ it follows that $\deg(\delta) = d_3$ and $\deg(\eta) = d_2$.

Similarly, $\operatorname{Homgr}_{K^b(P_{SD(3C)_{II}})}(T_2,T_0) \cong k\langle -(rd_2+(r-1)d_3)\rangle$, and we deduce that $\deg(\lambda) = 2d_6 - d_3$. It follows from the relations of $SD(3H)^r$ that $\deg(\delta)+\deg(\lambda) = \deg(\gamma)+\deg(\beta)+\deg(\gamma)$. Combining this with the equations from the previous two paragraphs we get that $\deg(\gamma) = d_6$, and subsequently that $\deg(\beta) = 0$.

Therefore, the resulting graded quiver of $SD(3H)^r$ is given by

$$
\begin{tikzcd}
0\bullet \arrow[rr, bend left, "0"] & & \bullet 1 \arrow[ll, bend left, "d_6"] \arrow[dl, bend left, "d_3"] \\
& \overset{2}{\bullet} \arrow[ur, bend left, "d_2"] \arrow[ul, bend left, "2d_6-d_3"] &
\end{tikzcd}
$$

We proceed by transferring gradings from $SD(3H)^r$ to $SD(3C)^r_{2,I}$. Let us assume that $SD(3H)^r$ is graded and that in the quiver of $SD(3H)^r$, $\deg(\beta) = d_2$, $\deg(\gamma) = d_3$, $\deg(\delta) = d_4$, $\deg(\eta) = d_5$ and $\deg(\lambda) = d_6$. Since the relations of $SD(3H)^r$ are homogeneous the following equations hold

$$
\begin{aligned}
d_2 + 2d_3 &= d_4 + d_6, \\
rd_5 + (r-1)d_4 &= d_2 + d_6.
\end{aligned}
$$

The graded radical layers of the projective indecomposable $SD(3H)^r$-modules are:

$$
\begin{array}{ccccc}
 & & S_0 & & \\
d_6 & S_2 & & S_1 & d_3 \\
 & & & S_0 & d_2+d_3 \\
d_4+d_6 & & S_1 & & \\
2d_2+2d_3 & & S_0 & &
\end{array}
\quad , \quad
\begin{array}{ccccc}
 & & S_2 & & \\
 & & S_1 & & d_4 \\
 & & & S_2 & d_4+d_5 \\
 & & & S_1 & 2d_4+d_5 \\
d_2+d_4 & S_0 & & \vdots & \\
 & & & S_2 & (r-1)(d_4+d_5) \\
 & & & S_1 & rd_4+(r-1)d_5 \\
 & & S_2 & & 2d_2+2d_3
\end{array}
\quad ,
$$

$$
\begin{array}{ccccccc}
 & & & & S_1 & & \\
 & & & & & S_2 & d_5 \\
 & & & & & S_1 & d_4+d_5 \\
 & & & & & \vdots & \\
d_2 & & S_0 & & \oplus & S_2 & (r-1)d_5+(r-2)d_4 \\
d_2+d_3 & S_1 & & & & S_1 & (r-1)(d_4+d_5) \\
2d_2+d_3 & S_0 & \oplus & S_2 & & & d_2+d_6 \\
 & & & S_1 & & & 2d_2+2d_3
\end{array}
\quad .
$$

A tilting complex that tilts from $SD(3H)^r$ to $SD(3C)^r_{2,I}$ is given by the direct sum $T := T_0 \oplus T_1 \oplus T_2$, where T_0 is the stalk complex with P_1 in degree 0, T_1 is the stalk complex with P_0 in degree 0, and T_2 is given by

$$
0 \longrightarrow P_1 \xrightarrow{\delta} P_2\langle d_4 \rangle \longrightarrow 0 \ ,
$$

with non-zero terms in degrees 0 and 1.

From $\mathrm{Endgr}_{K^b(P_{SD(3H)^r})}(T_1) \cong k\langle 0\rangle \oplus k\langle -(d_2+d_3)\rangle \oplus k\langle -(2d_2+d_3)\rangle$ it follows that $\deg(\beta)+\deg(\gamma) = d_2+d_3 = \deg(\eta)+\deg(\delta)$ in the quiver of $SD(3C)^r_{2,I}$. Similarly, from

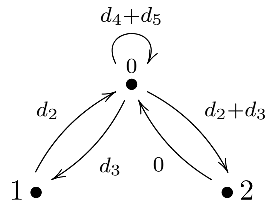
$$\mathrm{Endgr}_{K^b(P_{SD(3H)^r})}(T_0) \cong \bigoplus_{t=0}^{r} k\langle -t(d_4+d_5)\rangle \oplus k\langle -(d_2+d_3)\rangle$$

it follows that $\deg(\rho) = d_4+d_5$. We easily calculate that

$$\mathrm{Homgr}_{K^b(P_{SD(3H)^r})}(T_0, T_1) \cong k\langle -d_3\rangle \oplus k\langle -(2d_3+d_2)\rangle.$$

Therefore, $\deg(\gamma) = d_3$, and subsequently $\deg(\beta) = d_2$. Also, we have $\mathrm{Homgr}_{K^b(P_{SD(3H)^r})}(T_1, T_2) \cong \mathrm{Homgr}_{SD(3H)^r}(P_0, \ker(\delta)) \cong k\langle -(2d_2+d_3)\rangle$, which implies that $\deg(\beta\delta) = 2d_2+d_3$. Hence, $\deg(\delta) = d_2+d_3$ and $\deg(\eta) = 0$.

The resulting graded quiver of $SD(3C)^r_{2,I}$ is given by



We see that for each semidihedral block the resulting grading obtained by transfer of gradings via derived equivalences is a non-trivial grading for an appropriate choice of the homogeneous degrees of the arrows of an appropriate quiver. Thus, we have proved Theorem 1.2.

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