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A COMMON FIXED POINT THEOREM IN A MENGER SPACE USING WEAK COMPATIBILITY

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Abstract. A common fixed point theorem is established for four self maps on a complete Menger space assuming that a pair of maps has commmon fixed point and other pair is weakly compatible.

1. Introduction

Jungck ([1]) proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. Consequently, he introduced the notion of compatibility and established various fixed point theorems. Jungck and Rhodes ([2]) introduced the notion of weak compatibility which is a generalization of compatibility and considered the corresponding fixed point results. Mishra ([3]) established a fixed point result in a Menger space using compatibility. We generalize and extended this result using weak compatibility. The claim is also supported by an example.

2. Preliminaries

We take the standard definitions and results given in Schweizer and Sklar ([4]). We mainly use the following results in the subsequent section.

2.1. Result ([4]). Let $\{x_n\}(n = 0, 1, 2, \dots)$ be a sequence in a Menger space $(X, F, *)$, where $*$ is continuous and $x*x \geqslant x$ for all $x \in [0, 1]$. If there is a $k \in (0, 1)$ such that

$$F_{x_n,x_{n+1}}(kt) \geqslant F_{x_{n-1},x_n}(t)$$

for all $t > 0$ and $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.

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2.2. Result ([5]). Let $(X, F, *)$ be a Menger space. If there is a $k \in (0,1)$ such that

$$F_{x,y}(kt) \geqslant F_{x,y}(t)$$

for all $x, y \in X$ and $t > 0$, then $y = x$.

3. Main Result

We state the Theorem of Mishra ([3]).

Theorem 3.1. *Let A, B, S and T be self maps of a complete Menger space (X, F, t) with continuous t-norm and $t(x,x) \geqslant x$ for all $x \in [0,1]$, satisfying:*

(3.1.1) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$,*

(3.1.2) *for all $x, y \in X$, $u > 0$ and $\alpha \in (0,2)$ and for some $k \in (0,1)$*

$$F_{Ax,By}(ku) \geqslant t(F_{Ax,Sx}(u), t(F_{By,Ty}(u), t(F_{Ax,Ty}(\alpha u), F_{By,Sx}((2-\alpha)u)))),$$

(3.1.3) *the pairs $\{A, S\}$ and $\{B, T\}$ are compatible,*

(3.1.4) *S and T are continuous.*

Then A, B, S and T have a unique common fixed point in X.

Now, we prove the following generalization.

Theorem 3.2. *Let A, B, S and T be self mappings on a complete Menger space $(X, F, *)$, where $*$ is a continuous t-norm such that $u * u \geqslant u$, for all $u \in [0,1]$, satisfying:*

(3.2.1) *$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;*

(3.2.2) *either*

(i) A & S have a common fixed point and $\{B, T\}$ is weakly compatible

or

(ii) B & T have a common fixed point and $\{A, S\}$ is weakly compatible;

(3.2.3) *there is a $k \in (0,1)$ such that*

$$F^m_{Ax,By}(ku) \geqslant F^m_{Ax,Sx}(u) * F^m_{By,Ty}(u) * F^m_{Sx,Ty}(u) * F_{Ax,Ty}(\alpha u) * F_{By,Sx}((2-\alpha)u)$$

for all $x, y \in X$, for all $u > 0$, for all $\alpha \in (0,2)$ and for some positive integer m.

Then A, B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$. By virtue of (3.2.1) we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{2n} = T_{2n+1} = y_{2n}(say)$$

$$and \quad Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}(say), for \quad n = 0, 1, 2, \ldots$$

Taking $x = x_{2n}$, $y = x_{2n+1}$ and $\alpha = 1-q$ with $q \in (0,1)$ in (3.2.3), for $n \geqslant 1$,we get that

$$F^m_{y_{2n},y_{2n+1}}(ku) \geqslant$$

$$F^m_{y_{2n},y_{2n-1}}(u) * F^m_{y_{2n+1},y_{2n}}(u) * F^m_{y_{2n-1},y_{2n}}(u) * F_{y_{2n},y_{2n}}((1-q)u) * F_{y_{2n+1},y_{2n-1}}((1+q)u)$$

Using the properties of F, viz.
$F_{x,z}(u+v) \geqslant F_{x,y}(u) * F_{y,z}(v)$, for all $x, y, z \in X$ & $u, v > 0$, $F_{x,y}(u) = F_{y,x}(u)$, $F_{x,x}(u) = 1$ for all $x, y \in X$ & $u > 0$ and that of $*$, we get that

$$F^m_{y_{2n},y_{2n+1}}(ku) \geqslant F^m_{y_{2n-1},y_{2n}}(u) * F^m_{y_{2n},y_{2n+1}}(u) * F_{y_{2n},y_{2n+1}}(qu).$$

As t-norm is continuous and F is left continuous, $q \to 1-0$, we get that

$$F^m_{y_{2n},y_{2n+1}}(ku) \geqslant F^m_{y_{2n-1},y_{2n}}(u) * F^m_{y_{2n},y_{2n+1}}(u).$$

$$\Rightarrow F_{y_{2n},y_{2n+1}}(ku) \geqslant F_{y_{2n-1},y_{2n}}(u) * F_{y_{2n},y_{2n+1}}(u).$$

Similarly, by taking $x = x_{2n+2}, y = x_{2n+1}$ and $\alpha = 1 + q$ with $q \in (0,1)$ in (3.2.3), we get that

$$F_{y_{2n+1},y_{2n+2}}(ku) \geqslant F_{y_{2n},y_{2n+1}}(u) * F_{y_{2n+1},y_{2n+2}}(u).$$

Thus, for any positive integer n, we have

$$F_{y_n,y_{n+1}}(ku) \geqslant F_{y_{n-1},y_n}(u) * F_{y_n,y_{n+1}}(u).$$

Consequently, $F_{y_n,y_{n+1}}(u) \geqslant F_{y_{n-1},y_n}(k^{-1}u) * F_{y_n,y_{n+1}}(k^{-1}u)$.
By repeated application of the above inequality to $F_{y_n,y_{n+1}}(k^{-l}u)$ etc. and using the properties of $*$, we get that $F_{y_n,y_{n+1}}(u) \geqslant F_{y_{n-1},y_n}(k^{-1}u) * F_{y_n,y_{n+1}}(k^{-l}u)$, for any positive integer l.
Further, $F_{y_n,y_{n+1}}(k^{-l}u) \to 1$ as $l \to \infty$ (since $k^l u \to \infty$); so we get that

$$F_{y_n,y_{n+1}}(u) \geqslant F_{y_{n-1},y_n}(k^{-1}u)$$

$\Rightarrow F_{y_n,y_{n+1}}(ku) \geqslant F_{y_{n-1},y_n}(u)$, for all positive integer n.
Now, by Result (2.1), follows that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there is a $z \in X$ such that $\{y_n\} \to z$. So, follow that $\{y_{2n}\} = \{Ax_{2n}\} = \{Tx_{2n+1}\} \to z$ and $\{y_{2n+1}\} = \{Bx_{2n+1}\} = \{Sx_{2n}\} \to z$.

Suppose (3.2.2)(i) holds; now there is a $v \in X$ such that $Av = Sv = v$.
Taking $x = v, y = x_{2n+1}$ and $\alpha = 1$ in (3.2.3) and using $Av = Sv$, we get that

$$F^m_{Av,y_{2n+1}}(ku) \geqslant F^m_{Av,Av}(u) * F^m_{y_{2n+1},y_{2n}}(u) * F^m_{Av,y_{2n}}(u) * F_{Av,y_{2n}}(u) * F_{y_{2n},Av}(u).$$

Now, as $n \to \infty$,we get that

$$F^m_{v,z}(ku) \geqslant F^m_{v,v}(u) * F^m_{z,z}(u) * F^m_{v,z}(u) * F_{v,z}(u) * F_{z,v}(u) \geqslant F^m_{v,z}(u).$$

By Result(2.2), we get that $v = z$ so $Az = Sz = z$.

Since $A(X) \subseteq T(X)$, there is a $w \in X$ such that $z = Tw$.
Taking $x = x_{2n}, y = w$ and $\alpha = 1$ in (3.2.3), we get that

$$F^m_{y_{2n},Bw}(ku) \geqslant F^m_{y_{2n},y_{2n-1}}(u) * F^m_{Bw,z}(u) * F^m_{y_{2n-1},z}(u) * F_{y_{2n},z}(u) * F_{Bw,y_{2n-1}}(u).$$

Now, as $n \to \infty$,we get that

$$F^m_{z,Bw}(ku) \geqslant F^m_{z,z}(u) * F^m_{Bw,z}(u) * F^m_{z,z}(u) * F_{z,z}(u) * F_{Bw,z}(u) \geqslant F^m_{z,Bw}(u).$$

By the Result(2.2), we get that $Bw = z(= Tw)$.

Since $\{B, T\}$ is weakly compatible, follows that $BTw = TBw$; i.e, $Bz = Tz$. Taking $x = x_{2n}, y = z$, $\alpha = 1$ and $Tz = Bz$ in (3.2.3), we get that

$F^m_{y_{2n},Bz}(ku) \geqslant F^m_{y_{2n},y_{2n-1}}(u) * F^m_{Bz,Bz}(u) * F^m_{y_{2n-1},Bz}(u) * F_{y_{2n},Bz}(u) * F_{Bz,y_{2n-1}}(u)$.

Now, as $n \to \infty$,we get that

$$F^m_{z,Bz}(ku) \geqslant F^m_{z,z}(u) * F^m_{Bz,Bz}(u) * F^m_{z,Bz}(u) * F_{z,Bz}(u) * F_{Bz,z}(u) \geqslant F^m_{z,Bz}(u).$$

So, we get that $Bz = z \Rightarrow Bz = Tz = z$. Thus $Az = Bz = Sz = Tz = z$.

Similarly in the case (3.2.2)(ii) we first get that $Bz = Tz = z$ and then $Az = Sz = z$.

Uniqueness:- Let $z^{'}$ be also a common fixed point for A, B, S and T. So, $Az^{'} = Bz^{'}Sz^{'} = Tz^{'} = z^{'}$.

Taking $x = z, y = z^{'}$ and $\alpha = 1$ in (3.2.3), we get that

$$F^m_{Az,Bz'}(ku) \geqslant F^m_{Az,Sz}(u) * F^m_{Bz',Tz'}(u) * F^m_{Sz,Tz'}(u) * F_{Az,Tz'}(u) * F_{Bz',Sz}(u)$$

$$i.e, F^m_{z,z'}(ku) \geqslant F^m_{z,z}(u) * F^m_{z',z'}(u) * F^m_{z,z'}(u) * F_{z,z'}(u) * F_{z',z}(u) \geqslant F^m_{z,z'}(u).$$

By Result (2.2), follows that $z^{'} = z$. Hence z is the unique common fixed point for A, B, S and T. $\square$

We support this by means of the following:

Example 3.1. $(X, F, *)$ is a Menger space, where $X = [0, 10)$ with the usual metric and $F : \mathbb{R} \to [0, 1]$ is defined by

$$F_{x,y}(u) = \frac{u}{u + |x - y|}$$

for all $x, y \in \mathbb{R}$, $u > 0$ and $*$ is the min t-norm, i.e, $a * b = min\{a, b\}$ for all $a, b \in [0, 1]$.

Let A, B, S and T be the self maps on X, defined by

$$A(x) = \begin{cases} 0 & \text{if } x \leqslant 9, \\ 1 & \text{if } x > 9. \end{cases}$$

$$S(x) = \begin{cases} 0 & \text{if } x \leqslant 9, \\ x^{\frac{1}{2}} & \text{if } x > 9. \end{cases}$$

$Bx = 0$ and $Tx = x$ for all $x \in X$.

Then, clearly A, B, S and T satisfy the hypothesis of Theorem(3.2) with $k \in [\frac{1}{2}, 1) \subset (0, 1)$.

For, when $x > 9$,

$$F^m_{Ax,By}(ku) = \Big(\frac{ku}{ku + 1}\Big)^m = \Big(\frac{u}{u + \frac{1}{k}}\Big)^m$$

and

$$F^m_{Ax,Sx}(u) = \Big(\frac{u}{u + (x^{\frac{1}{2}} - 1)}\Big)^m < \Big(\frac{u}{u + 2}\Big)^m.$$

So, in (3.2.3), L.H.S $\geqslant$ R.H.S when $\frac{1}{k} \leqslant 2$, that is $k \geqslant \frac{1}{2}$.
Clearly 0 is the unique common fixed point of A, B, S and T.

(Observe that S is not continuous on X.)

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