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GENERALIZATION OF SOME RESULTS TO QUASI-METRIC SPACES AND ITS APPLICATIONS

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Abstract. We establish convergence theorems of a sequence in quasi-metric spaces. As applications, some previous results are obtained from these theorems as special cases.

1. Introduction

Some convergence theorems of certain iterations to a fixed point for quasi-nonexpansive and generalized types of quasi-nonexpansive mappings have appeared (see, for example [**1**], [**3**], [**7**]).

Let X be a set. Following Waszkiewick ([**9**]), a *distance* on X is a map $d : X \times X \to [0, \infty)$. A pair (X, d) is called a *distance space*. If d satisfies the following conditions, for every $x, y, z \in X$,

(M_1) $\quad d(x, x) = 0$;
(M_2) $\quad d(x, y) = d(y, x) = 0 \Longrightarrow x = y$;
(M_3) $\quad d(x, y) \leqslant d(x, z) + d(z, y)$,

then it is called a *quasi-metric* (or simply q-metric) on X. If d satisfies (M_2) and (M_3), then d is said to be a *dislocated quasi-metric* (or simply dq-metric) on X. It is clear that if d satisfies $(M_1) - (M_3)$ and

(M_4) $\quad d(x, y) = d(y, x) \qquad \forall x, y \in X$,

then d is a *metric* on X.

Example 1.1. Let $X = R$ ($R :=$ the set of all real numbers) be endowed with the metric d defined by $d(x, y) = |x - y| \quad \forall x, y \in X$. We find that

$$d(kx, ky) = |kx - ky| = |k(x - y)| = |k||x - y| = |k|d(x, y),$$

for all $x, y \in X$ and for each $k \in R$.

If X is a vector space, then we can define a function $\|.\|$ from X into $[0, \infty)$. Consider the following properties of a function $\|.\|$, for all $x, y, z \in X$,

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(N_1) $\|x-y\| = \|y-x\| = 0 \Longrightarrow x = y$;
(N_2) $\|x-y\| \leqslant \|x-z\| + \|z-y\|$;
(N_3) $\|x-y\| = \|y-x\|$;
(N_4) $\|0\| = \|x-x\| = 0$;
If $\|.\|$ satisfies (N_1) and (N_2), then it is called a *dislocated quasi-norm* (or simply dq-norm) on X. If $\|.\|$ satisfies $(N_1)-(N_4)$ and the condition
(N_5) $\|\alpha(x-y)\| = |\alpha|\|x-y\| \quad \forall x, y \in X, \quad \forall \alpha \in R$,
then d is a *norm* on X. $(X, \|.\|)$ is called a dq-normed space (resp. q-normed space, normed space) where $\|.\|$ is a dq-norm (resp. q-norm, norm) on X.

It is clear that $norm \Longrightarrow q-norm$ and $q-norm \Longrightarrow dq-norm$. The reverse implications may not be true. One can deduce that the distance function induced by q-norm (resp. dq-norm) from the formula $d(x,y) = \|x-y\| \quad \forall x, y \in X$ is a q-metric (resp. dq-metric) on X.

Let D be a subset of a q-metric space (X,d) and $T : D \to X$ be any mapping. Assume that $F(T)$ is the set of all fixed points of T. For a given $x_0 \in D$, the sequence of iterate (x_n) is determined by
(I) $\quad x_n = T^n(x_{n-1}) = T^n(x_0), \qquad n \in N$
where N is the set of all positive integers.

If X is a normed space, D is a convex set and $T : D \to D$, Ishikawa ([**4**]) gave the following iteration
(II) $\quad x_n = T_{\lambda,\mu}(x_{n-1}) = T^n_{\lambda,\mu}(x_0), \quad T_{\lambda,\mu} = (1-\lambda)I + \lambda T[(1-\mu)I + \mu T]$,
for each $n \in N$, where $\lambda \in (0,1)$ and $\mu \in [0,1)$ When $\mu = 0$, it yields that $T_{\lambda,0} = (1-\lambda)I + \lambda T = T_\lambda$. Therefore, the iteration scheme (II) becomes $x_n = T_\lambda(x_{n-1}) = T^n_\lambda(x_0)$. This iteration is called Mann iteration ([**6**]).

Definition 1.1. ([**7**]) Let (X,d) be a metric space. A mapping $T : X \to X$ is called *quasi-nonexpansive* if for each $x \in D$ and for every $p \in F(T)$, $d(T(x),p) \leqslant d(x,p)$.

Definition 1.2. ([**1**]) Let (X,d) be a metric space. The map $T : D \to X$ is said to be *quasi-nonexpansive w.r.t.* $(x_n) \subseteq D$ if for all $n \in N \cup \{0\}$ and for every $p \in F(T)$, $d(x_{n+1},p) \leqslant d(x_n,p)$.

Lemma 2.1 in [**1**] stated that

quasi-nonexpansiveness $\Longrightarrow$ quasi-nonexpansiveness w.r.t.

$(T^n(x_0))$ (respectively, $(T^n_\lambda(x_0))$, $(T^n_{\lambda,\mu}(x_0))$) for each $x_0 \in D$. The reverse implications are not true (see, Example 2.1 in [**1**]). Also, the authors in [**1**] showed that the continuity of $T : D \to X$ leads to the closedness of $F(T)$ and the converse is not true (see, Example 2.2 in [**1**]).

Definition 1.3. ([**5**]) Let (X,d) be a metric space. The mapping $T : X \to X$ is called an *asymptotically regular at a point* $x_0 \in X$ if

$$\lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = 0.$$

Definition 1.4. ([**10**]) A sequence (x_n) in a dq-metric space (X, d) is said to be *convergent* to an $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$ or $\lim_{n\to\infty} d(x, x_n) = 0$.
In this case x is called a *limit* of (x_n) and we write $\lim_{n\to\infty} x_n = x$.

Definition 1.5. ([**10**]) A sequence (x_n) in a dq-metric space (X, d) is said to be *Cauchy* if for every $\epsilon > 0$ there is an $n_0 = n_0(\epsilon) \in N$ such that $\forall m, n \geqslant n_0$, $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$.

Replacing $d(x_m, x_n) < \epsilon$ or $d(x_n, x_m) < \epsilon$ in Definition 1.6 by

$$\max\{d(x_m, x_n), d(x_n, x_m)\} < \epsilon,$$

the sequence (x_n) in a dq-metric space (X, d) is called *bi-Cauchy* (see [**8**]).

Definition 1.6. ([**10**]) (X, d) is said to be a *complete dq-metric space* if every Cauchy sequence in X converges to a point in X.

Lemma 1.1. ([**10**]) *Let (X, d) be a dq-metric space. Then every convergent sequence in X is Cauchy.*

It is obvious that the converse of Lemma 1.1 may not be true.

Lemma 1.2. ([**10**]) *Let (X, d) be a dq-metric space. If (x_n) is a sequence in X converging to $x \in X$, then every subsequence of (x_n) converges to x.*

Lemma 1.3. ([**10**]) *dq-limits in a dq-metric space are unique.*

Remark 1.1. (1) Since any q-metric is dq-metric, then Definitions 1.5-1.7 and Lemmas 1.1-1.3 remain valid in q-metric spaces.
(2) For any subset A of a metric space (X, d), $d(x, A) = 0$ if and only if $x \in \overline{A}$ where $\overline{A}$ is the closure of the set A. This fact is valid in q-metric spaces. If A is a closed subset of a metric or a q-metric space, then $d(x, A) = 0$ if and only if $x \in A$.

The aim of our study is to establish some convergence theorems of a sequence in complete q-metric spaces. These theorems include some results of Ahmed and Zeyada ([**1**]), and Petryshyn and Williamson ([**7**]) as special cases.

2. Main Results

First we present Definitions 1.3 and 1.4 in q-metric spaces.

Definition 2.1. Let (X, d) be a q-metric space. The map $T : D \to X$ is said to be *quasi-nonexpansive w.r.t.* $(x_n) \subseteq D$ if for all $n \in N \cup \{0\}$ and for every $p \in F(T)$, $d(x_{n+1}, p) \leqslant d(x_n, p)$.

Definition 2.2. Let (X, d) be a q-metric space. The mapping $T : X \to X$ is called *asymptotically regular at a point* $x_0 \in X$ if $\lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = 0$.

We state and prove the following lemma.

Lemma 2.1. *Let (X, d) be a q-metric space and (x_n) be a sequence in $D \subseteq X$. Assume that $T : D \to X$ is a mapping with $F(T) \neq \phi$. If T is quasi-nonexpansive w.r.t. (x_n), then $(d(x_n, F(T)))$ is a monotonically decreasing sequence in* $[0, \infty)$.

Proof. Since T is quasi-nonexpansive w.r.t. (x_n), then

$$d(x_{n+1}, p) \leqslant d(x_n, p) \tag{1}$$

for all $n \in N \cup \{0\}$ and for every $p \in F(T)$. From (1), taking the infimum over $p \in F(T)$, we get that $d(x_{n+1}, F(T)) \leqslant d(x_n, F(T))$ for all $n \in N \cup \{0\}$. Hence, $(d(x_n, F(T)))$ is a monotonically decreasing sequence in $[0, \infty)$. □

The following example shows that the converse of Lemma 2.1 may not be true.

Example 2.1. Let $X = [0, 1]$ be endowed with the Euclidean metric d. We define the map $T : X \to X$ by $T(x) = \frac{3}{4}x^2 + \frac{1}{4}x$ for each $x \in X$. Assume that $x_n = \frac{1}{n}$ $\forall n \in N - \{1, 2, 3\}$. Then $F(T) = \{0, 1\}$ and the sequence $(d(x_n, F(T))) = (\frac{1}{n})$ is monotonically decreasing in $[0, \infty)$. But, T is not quasi-nonexpansive w.r.t. (x_n) (Indeed, there exists $1 \in F(T)$ such that $\forall n \in N - \{1, 2, 3\}$, $d(x_{n+1}, 1) > d(x_n, 1)$).

In view of Lemma 2.1 in [**1**], we state the following lemma and its proof is similar to the proof of Lemma 2.1 ([**1**]).

Lemma 2.2. *Let (X, d) be a q-metric space and (x_n) be a sequence in $D \subseteq X$. Assume that $T : D \to X$ is quasi-nonexpansive w.r.t. (x_n) with $F(T) \neq \phi$ satisfying $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Then, (x_n) is a Cauchy sequence.*

Now, we give the main theorem in the following way.

Theorem 2.1. *Let (x_n) be a sequence in a subset D of a q-metric space (X, d) and $T : D \to X$ be a map such that $F(T) \neq \phi$. Then*
(a) $\lim_{n\to\infty} d(x_n, F(T)) = 0$ if (x_n) converges to a unique point in $F(T)$;
(b) (x_n) converges to a unique point in $F(T)$ if $\lim_{n\to\infty} d(x_n, F(T)) = 0$, $F(T)$ is a closed set, T is quasi-nonexpansive w.r.t. (x_n) and X is complete.

Proof. (a) Since (x_n) converges to a unique point in $F(T)$, then there exists a unique point $p \in F(T)$ such that $\lim_{n\to\infty} d(x_n, p) = 0$. Then, we obtain from the continuity of the map $x \to d(x, F(T))$ that

$$\lim_{n\to\infty} d(x_n, F(T)) = d(\lim_{n\to\infty} x_n, F(T))$$
$$= inf_{q\in F(T)} d(\lim_{n\to\infty} x_n, q) = inf_{q\in F(T)} d(p, q) = d(p, p) = 0.$$

(b) From Lemma 2.2 and the completeness of X, then there exists a unique point $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Since the map $x \to d(x, F(T))$ is continuous (see, [2, p. 13]), then

$$0 = \lim_{n\to\infty} d(x_n, F(T)) = d(\lim_{n\to\infty} x_n, F(T)) = d(p, F(T)).$$

Therefore, we obtain from the closedness of $F(T)$ that $p \in F(T)$. □

Also, we state and prove the following theorem.

Theorem 2.2. *Let (x_n) be a sequence in a subset D of a complete q-metric space (X, d) and $T : D \to X$ be a map such that $F(T) \neq \phi$ is a closed set. Assume that*
(i) T is quasi-nonexpansive w.r.t. (x_n);

(ii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$*;*
(iii) if the sequence (y_n) *satisfies* $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$*, then* $\liminf_n d(y_n, F(T)) = 0$ *or* $\limsup_n d(y_n, F(T)) = 0$*.*
Then (x_n) *converges to a unique point in* $F(T)$*.*

Proof. From Lemma 2.1 and the boundedness from below by zero of the sequence $(d(x_n, F(T)))$, then $\lim_{n\to\infty} d(x_n, F(T))$ exists. So, we obtain from conditions (ii) and (iii) that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hence, by Theorem 2.1(a), the sequence (x_n) converges to a unique point in $F(T)$. □

Remark 2.1. Theorem 2.1 (resp. Theorem 2.2) generalizes and improves Theorem 2.1 in [**1**] (resp. Theorem 2.2 in [**1**]) since
(I) X is a q-metric space instead of X is a metric space;
(II) The condition " $F(T)$ is closed set" is used only in the item (b) of Theorem 2.1 but not in item (a) of this theorem.

Finally, we establish another consequence of Theorem 2.1 as follows.

Theorem 2.3. *Let* (x_n) *be a sequence in a subset* D *of a complete q-metric space* (X, d)*. Furthermore, let* $T : D \to X$ *be a quasi-nonexpansive mapping w.r.t.* (x_n) *such that* $F(T) \neq \phi$ *is a closed set. Assume that*
(iii)$'$ *the sequence* (x_n) *contains a convergent subsequence* (x_{n_j}) *converging to* $x^* \in D$ *such that there exists a continuous mapping* $S : D \to D$ *satisfying* $S(x_{n_j}) = x_{n_{j+1}}$ $\forall j \in N$ *and* $d(S(x^*), p) < d(x^*, p)$ *for some* $p \in F(T)$*.*
Then $x^* \in F(T)$ *and* $\lim_{n\to\infty} x_n = x^*$*.*

Proof. From Lemma 2.1, the quasi-nonexpansiveness of T w.r.t. (x_n) implies that $\lim_{n\to\infty} d(x_n, F(T))$ exists, say equal $r \in [0, \infty)$. Suppose that x^* doesn't belong to $F(T)$. So, we have from (iii)$'$ that for some $p \in F(T)$,

$$d(x^*, p) > d(S(x^*), p) = d(S(\lim_{j\to\infty} x_{n_j}), p) =$$
$$d(\lim_{j\to\infty} S(x_{n_j}), p) = d(\lim_{j\to\infty} x_{n_j+1}, p) = d(x^*, p).$$

This contadiction implies that $x^* \in F(T)$. Then,

$$r = \lim_{n\to\infty} d(x_n, F(T)) = \lim_{j\to\infty} d(x_{n_j}, F(T)) =$$
$$d(\lim_{j\to\infty} x_{n_j}, F(T)) = d(x^*, F(T)) = 0.$$

From Theorem 2.1 (b), we obtain that $\lim_{n\to\infty} x_n = x^*$. □

Corollary 2.1. *For each* $x_0 \in D$*, let* $(T^n(x_0))$ *be a sequence in a subset* D *of a complete q-metric space* (X, d)*. Furthermore, let* $T : D \to X$ *be a quasi-nonexpansive mapping w.r.t.* $(T^n(x_0))$ *such that* $F(T) \neq \phi$ *is a closed set. Assume that*
(iii)$'$ *the sequence* $(T^n(x_0))$ *contains a convergent subsequence* $(T^{n_j}(x_0))$ *converging to* $x^* \in D$ *such that there exists a continuous mapping* $S : D \to D$ *satisfying* $S(T^{n_j}(x_0)) = T^{n_j+1}(x_0)$ $\forall j \in N$ *and* $d(S(x^*), p) < d(x^*, p)$ *for some* $p \in F(T)$*.*
Then $x^* \in F(T)$ *and* $\lim_{n\to\infty} T^n(x_0) = x^*$*.*

Corollary 2.2. *For each $x_0 \in D$, let $(T_\lambda^n(x_0))$ be a sequence in a subset D of a complete q-normed space (X,d). Furthermore, let $T : D \to X$ be quasi-nonexpansive mapping w.r.t. $(T_\lambda^n(x_0))$ such that $F(T) \neq \phi$ is a closed set. Assume that*

(iii)$'$ *the sequence $(T_\lambda^n(x_0))$ contains a convergent subsequence $(T_\lambda^{n_j}(x_0))$ converging to $x^* \in D$ such that there exists a continuous mapping $S : D \to D$ satisfying $S(T_\lambda^{n_j}(x_0)) = T_\lambda^{n_j+1}(x_0) \quad \forall j \in N$ and $\|S(x^*) - p\| < \|x^* - p\|$ for some $p \in F(T)$. Then $x^* \in F(T)$ and $\lim_{n\to\infty} T_\lambda^n(x_0) = x^*$.*

Corollary 2.3. *For each $x_0 \in D$, let $(T_{\lambda,\mu}^n(x_0))$ be a sequence in a subset D of a complete q-normed space (X,d). Furthermore, let $T : D \to X$ be quasi-nonexpansive mapping w.r.t. $(T_{\lambda,\mu}^n(x_0))$ such that $F(T) \neq \phi$ is a closed set. Assume that*

(iii)$'$ *the sequence $(T_{\lambda,\mu}^n(x_0))$ contains a convergent subsequence $(T_{\lambda,\mu}^{n_j}(x_0))$ converging to $x^* \in D$ such that there exists a continuous mapping $S : D \to D$ satisfying $S(T_{\lambda,\mu}^{n_j}(x_0)) = T_{\lambda,\mu}^{n_j+1}(x_0) \quad \forall j \in N$ and $\|S(x^*) - p\| < \|x^* - p\|$ for some $p \in F(T)$. Then $x^* \in F(T)$ and $\lim_{n\to\infty} T_{\lambda,\mu}^n(x_0) = x^*$.*

Remark 2.2. Theorem 1.3 in [**7**] is a special case of Corollary 2.1 for some reasons. These reasons are

(1) the closedness of D is superfluous;

(2) $F(T)$ is closed instead of T is continuous;

(3) X is a complete q-metric space instead of X is a Banach space;

(4) T is quasi-nonexpansive w.r.t. $(T^n(x_0))$ in lieu of T is quasi-nonexpansive;

(5) the conditions (1.6) and (1.7) in Theorem 1.3 in [**7**] are generalized to the condition (iii)$'$ in Corollary 2.1.

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