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EQUI-INTEGRATY PARTITIONS IN GRAPHS

Sundareswaran, R. and Swaminathan, V.

Abstract. C.A. Barefoot, et. al. introduced the concept of the integrity of a graph. It is an useful measure of vulnerability and it is defined as follows $I(G) = min\{|S| + m(G - S) : S \subset V(G)\}$, where $m(G - S)$ denotes the order of the largest component in $G - S$. The integrity of the set S is defined as $|S| + m(G - S)$ and is denoted by I_S, where $m(G - S)$ denotes the order of maximum component in $G - S$. A partition of $V(G)$ into subsets $V_1, V_2, \cdots, V_t$ such that $I_{V_i}, 1 \leqslant i \leqslant t$ is a constant is called equi-integrity partition of G. The maximum cardinality of such a partition is called equi-integrity partition number of G and is denoted by $EI(G)$. Since $V(G)$ itself is an equi-integrity partition of G, the existence of EI-partition is guaranteed. In this paper, a study of this new parameter is initiated.

Keywords: Integrity, Equi-Integrity Partitions

1. Introduction

The stability of a communication network is of prime importance for network designers. In an analysis of the vulnerability of a communication network to disruption, two quantities that come to our mind are the number of elements that are not functioning and the size of the largest remaining sub network within which mutual communications can still occur. In adverse relationship, it would be desirable for an opponent's network to be such that the two quantities can be made simultaneously small. In articles of C.A. Barefoot, R.Entriger and H.Swart ([**1**]) and G. Chartrand, S.F. Kapoor, T.A. McKee and O.R. Oellermann ([**4**]) and (See, also, W.D.Goddard [**2**] and K.S. Bagga, L.W. Beineke, W.D. Goddard, M.J. Lipman and R.E. Pippert [**3**]) introduced the concept of the integrity of a graph. It is an useful measure of vulnerability and it is defined as follows $I(G) = min\{|S| + m(G - S) : S \subset V(G)\}$,

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where $m(G-S)$ denotes the order of the largest component in $G-S$. Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also the damage that has been caused. A partition of $V(G)$ into subsets $V_1, V_2, \cdots, V_t$ such that $I_{V_i}, 1 \leqslant i \leqslant t$ is a constant is called equi-integrity partition of G. The maximum cardinality of such a partition is called equi-integrity partition number of G and is denoted by $EI(G)$. Since $V(G)$ itself is an equi-integrity partition of G, the existence of EI-partition is guaranteed.

This new parameter is related to the connectivity of the graph. If G has no cut vertices, then $EI(G)$ is maximum, namely the order of the graph. If a graph has more cut vertices such that the connected components resulting out of the removal of a cut vertex are more in number and have smaller orders, then $EI(G)$ becomes small. Thus, this parameter has relationship with connected graphs of connectivity one.

2. Equi-Integrity Partitions of graphs

Definition 2.1. ([**1**]) A set of vertices S in a graph G is an I-set of G if $|S| + m(G-S) = I(G)$.

Definition 2.2. For a subset S of $V(G)$, let $I_s = |S| + m(G-S)$, where $m(G-S)$ denotes the order of the largest component in $G-S$.

Definition 2.3. A partition of $V(G)$ into subsets $V_1, V_2, \cdots, V_t$ such that $I_{V_i}, 1 \leqslant i \leqslant t$ is a constant is called equi-integrity partition of G. The maximum cardinality of such a partition is called equi-integrity partition number of G and is denoted by $EI(G)$.

Remark 2.1. Since $V(G)$ itself is an equi-integrity partition of G, the existence of EI-partition is guaranteed.

Theorem 2.1. *Let G be a nontrivial connected graph with order n. Then $EI(G) = n$ if and only if G has no cut vertex.*

Proof. Suppose G is a nontrivial connected graph without cut vertex. Then $EI(G) = n$.

Conversely, suppose G is a nontrivial connected graph. Then G has at least two vertices which are not cut vertices. For such a vertex say $u, I_u = n$. If G has a cut vertex say v. Then $I_v = 1 + m(G-v) < 1 + n - 1 = n$, a contradiction . □

Remark 2.2. It can be easily shown that $EI(K_n) = n, EI(K_{m,n}) = m + n, EI(W_{1,n}) = n+1, EI(C_n) = n$.

Theorem 2.2. $EI(P_n) = \lceil \frac{n}{2} \rceil$.

Proof. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$.

If n is odd and $n = 2k+1$, then$\{\{v_1, v_{k+1}\}, \{v_2, v_{k+3}\}, \cdots, \{v_k, v_{2k+1}\}, \{v_{k+2}\}\}$ is a EI-partition with $I_{V_i} = k+2$, for all $i, 1 \leqslant i \leqslant k+1$. Therefore, $EI(P_n) \geqslant \frac{n+1}{2}$. Suppose $EI(P_n) > \frac{n+1}{2} = k+1$. Then any maximum EI-partition has at least three singletons with equal integrity, a contradiction, since P_n has at most two singletons have same integrity. Therefore, $EI(P_n) = \frac{n+1}{2}$.

If n is even and $n = 2k$, then $\{\{v_1, v_{k+1}\}, \{v_2, v_{k+2}\}, \cdots, \{v_k, v_{2k}\}\}$ is a EI-partition with $I_{V_i} = k + 1$, for all $i, 1 \leqslant i \leqslant k$. Therefore, $EI(P_n) \geqslant \frac{n}{2}$. Suppose $EI(P_n) > n/2 = k$. Then any maximum EI-partition has at least two singletons. If there are three or more singletons, we get a contradiction. Therefore, any maximum EI-partition contains exactly two singletons and the remaining are doubletons. It can be easily verified that in a such a partition, the set may not have equal integrity. Therefore, $EI(P_n) \leqslant \frac{n}{2}$. Therefore, $EI(P_n) = \lceil \frac{n}{2} \rceil$. □

Definition 2.4. Let G be graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$. The Mycielski transformation of G, denoted $\mu(G)$, has for its vertex set, the set $\{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, z\}$. As for adjacency, x_i is adjacent with x_j in $\mu(G)$ if and only if v_i is adjacent with v_j in G, x_i is adjacent with y_j in $\mu(G)$ if and only if v_i is adjacent with v_j in G, and y_i is adjacent with z in $\mu(G)$ for all $i \in \{1, 2, \cdots, n\}$.

Corollary 2.1. *If G is any connected graph of order n, then $EI(\mu(G)) = |V(\mu(G))|$, since $(\kappa(\mu(G))) \geqslant 2$.*

Theorem 2.3. *Let G be a connected graph of order n without cut vertices. Attach one pendent vertex each at k of the vertices of G. Let H be the resulting graph. Then $EI(H) = n$.*

Proof. Let $\{u_1, u_2, \cdots, u_n\}$ be the vertex set of G. Let $\{u_1, u_2, \cdots, u_k\}$ be the set of vertices of G at which pendent vertices $v_1, v_2, \cdots, v_k$ are attached. Then $\{\{u_1, v_1\}, \{u_2, v_2\}, \cdots, \{u_k, v_k\}, \cdots, \{u_{k+1}\}, \cdots, \{u_n\}\}(= \{\{V_1\}, \cdots, \{V_n\}\})$ is a maximum EI-partition of H and $I_{V_i} = n + k$ for all $i, 1 \leqslant i \leqslant n$. Therefore, $EI(H) = n$. □

Theorem 2.4. $EI(K_{1,n}) = 2$.

Proof. Let $V(K_{1,n}) = \{u, v_1, v_2, \cdots, v_n\}$ and u be the vertex of degree n. Let $\{V_1, V_2, \cdots, V_t\}$ be a maximum EI-partition of $V(K_{1,n})$. Suppose $u \in V_1$. Let $I_{V_1} = i + 2$, where $|V_1| = i + 1$. For any $V_j, 2 \leqslant j \leqslant t$, $I_{V_j} = n + 1$. Since $I_{V_1} = I_{V_j}$ Therefore, $i = n - 1$ and hence $|V_1| = n$. Hence $t = 2$. Therefore, $|V_2| = 1$. □

Theorem 2.5. *Let G be a star with at least three pendent vertices. Let H be the graph obtained from G in which each edge of G is subdivided exactly once. Then $EI(H) = 2$ or 3.*

Proof. Let G be a star with at least three pendent vertices. Let H be the graph obtained from G in which each edge of the G is subdivided exactly once. Let u be the center vertex of H and $x_1, x_2, \cdots ., x_n$ be the vertices of degree two in H and $y_1, y_2, \cdots, y_n$ be the pendent vertices in H. Let $V_1 = \{u, y_1, y_2, \cdots, y_{n-1}\}$ and $V_2 = \{x_1, x_2, \cdots, x_n, y_n\}$. $I_{V_1} = I_{V_2} = n + 2$. $V_1 \cup V_2 = V(H)$ and $V_1 \cap V_2 = \emptyset$. Therefore, $EI(H) \geqslant 2$. Suppose $\{V_1, V_2, \cdots, V_k\}$ be an EI-partition of maximum cardinality of G with $k \geqslant 3$. Without loss of generality, let $u \in V_1$. Let $|V_1| = a$ and $|V_2| = b$. If $a \geqslant n+1$, then $I_{V_1} = a+1$ or $a+2$. If $a < n+1$, then $I_{V_1} = a+2$.

Case (I) Let $a \geqslant n+1$ and $b \geqslant n$.Then $|V_1|+|V_2| \geqslant 2n+1$. $V_3 = V_4 = \cdots = V_k = \emptyset$, a contradiction, since $k \geqslant 3$.

Case (II) Let $a < n+1$ and $b \geqslant n$.

Subcase (i): Suppose V_2 contains $x_1, x_2, \cdots, x_n$. Then $V_3 \subseteq \{y_1, y_2, \cdots, y_{n-1}\}$. Therefore, $I_{V_2} = 2n+1 = I_{V_1} = a+2 \Longrightarrow a = 2n-1 < n+1$. That is, $n < 2$, contradiction.

Subcase (ii): Suppose V_2 contains $y_1, y_2, \cdots, y_n$. Then $V_3 \subseteq \{x_1, x_2, \cdots, x_n\}$. Let $|V_3| = c$ (say). Then $I_{V_3} = c+2(n-c)+1 = 2n-c+1$. But $I_{V_2} = 2n+1 = I_{V_3}$. Therefore, $2n-c+1 = 2n+1 \Longrightarrow c = 0$, a contradiction.

Subcase (iii): Suppose V_2 contains $\alpha_1, x_i's, \beta_1, y_j's$ for which $x_j \in V_2$ and $\beta_2, y_j's$ for which $x_j \notin V_2$. Therefore $\alpha_1+\beta_1+\beta_2 = |V_2| = b \geqslant n$. There are $n-\alpha_1, x_i's$ in $V - V_2$ out of which $n-\alpha_1-\beta_2$ are pairs of x_i, y_i and $\beta_2, x_i's$. Therefore, $I_{V_2} = 2(n-\alpha_1-\beta_2)+\beta_2+1+b = 2n-\alpha_1+1+\beta_1$.

Suppose V_3 contains $\alpha_1', x_i's, \beta_1', y_j's$ for which $x_j \in V_3$ and $\beta_2', y_j's$ for which $x_j \notin V_3$. Then, $I_{V_3} = 2n-\alpha_1'+1+\beta_1'$. $I_{V_1} = I_{V_2} = I_{V_3} = a+2$. Therefore, $a = 2n-\alpha_1'+1+\beta_1'$. Since $a < n+1$, we get that $2n-\alpha_1'+1+\beta_1' < n+1$. Therefore, $n-\alpha_1'+1+\beta_1'' < 2$. Thus, $n-\beta_1' < \alpha_1'+2$.

Adding $\beta_1'+\beta_2'$ to both sides, $n+2\beta_1'+\beta_2' < \alpha_1'+\beta_1'+\beta_2'+2 = c+2 = (2n+1-a-b)+2$. Therefore, $2\beta_1'+\beta_2' < n+3-a-b$. Since $b \geqslant n$, $n-b \leqslant .0$. Therefore, $2\beta_1'+\beta_2' < 3-a$. That is, $2\beta_1'+\beta_2' \leqslant 2-a$. Since $a \geqslant 1, 2-a \leqslant 1$. Therefore, $2\beta_1'+\beta_2' \leqslant 1$.

Subsubcase(i): $2\beta_1'+\beta_2' = 0$. Therefore, $\beta_1' = 0 = \beta_2'$. Thus, $2\beta_1'+\beta_2' \leqslant 2-a$. This implies that $a \leqslant 2$. Let $a = 2$. $I_{V_3} = 2n-\alpha_1'+1 = I_{V_1} = a+2 = 4$. Therefore, $\alpha_1' = 2n-3$. Therefore, $|V_3| = 2n-3, |V_1| = 2$. Therefore,$|V_2| = 2 \geqslant n$. That is, $n \leqslant 2$, a contradiction. Let $a = 1$. $I_{V_3} = 2n-\alpha_1'+1 = I_{V_1} = a+2 = 3$. Therefore, $\alpha_1' = 2n-2$. Therefore, $|V_3| = 2n-2, |V_1| = 1$. Therefore, $|V_2| = 2 \geqslant n$. That is, $n \leqslant 2$, a contradiction.

Subsubcase(ii): $2\beta_1'+\beta_2' = 1$. Since $2\beta_1'+\beta_2' \leqslant 2-a$ we get that $1 \leqslant 2-a$ implies that $a \leqslant 1$. Therefore, $a = 1$. Since $\beta_1'+\beta_2' = 1$, we get that $\beta_1' = 0, \beta_2' = 1$. $I_{V_3} = 2n-\alpha_1'+\beta_1'+1 = 2n-\alpha_1'+1 = I_{V_1} = a+2 = 3$. Therefore, $|V_3| = \alpha_1'+\beta_1'+\beta_2' = 2n-1$. $|V_2| = 1 = b \geqslant n$. Therefore, $n \leqslant 1$, a contradiction.

Case (III): Let $a \geqslant n+1$ and $b < n$.

Subcase (i): $I_{V_1} = a+1$.

Subsubcase (i): Suppose V_1 contains $x_1, x_2, \cdots, x_n$. Then $V_2 \subseteq \{y_1, y_2, \cdots, y_n\}$. Therefore, $I_{V_2} = 2n+1 = I_{V_1} = a+1 \Longrightarrow a+1$. Therefore, $a = 2n.V_3 = \emptyset$, contradiction.

Subsubcase (ii): Since $I_{V_1} = a+1$, for any $x_i, y_i; 1 \leqslant i \leqslant n$, at least one of x_i, y_i belongs to V_1. Therefore, $\beta_1' = \beta_1 = 0$ Therefore, $I_{V_2} = 2n-\alpha_1+1 = I_{V_3} = 2n-\alpha_1'+1$.

Let V_1 contains $t, y_j's$, where $t \leqslant n-1$. Let without loss of generality, $y_1, y_2, \cdots, y_t \in V_1$. Then V_1 contains $x_{t+1}, x_{t+2}, \cdots, x_n$. Suppose $V_r = \{x_i\}$for some $i, 1 \leqslant i \leqslant t$. Then $I_{V_r} = 2n = a+1$. Therefore,$a = 2n-1$. Thus, there are exactly three sets V_1, V_2, V_3 such that V_2 and V_3 contain exactly one x_i and $I_{V_1} = I_{V_2} = I_{V_3} = 2n$. Note that, since $I_{V_1} = 2n$, V_2 or V_3 can not contain a single y_j. If $V_r = \{y_j\}$ for some $j, 1 \leqslant j \leqslant t$. Then $I_{V_r} = 2n + 1$. Therefore, 2n + 1 = a + 1. Therefore, a = 2n. Thus, there are exactly two sets V_1, V_2 with $I_{V_1} = I_{V_2} = 2n + 1$

Suppose$V_r = \{x_{i1}, x_{i2}\}$. Then $I_{V_r} = 2n - 1 = a + 1$. Therefore, $a = 2n - 2$. Since $|V_1| + |V_r| = 2n$, there is exactly one set say V_3 which is a singleton. If $V_3 = \{x_{i3}\}$, then $I_{V_3} = 2n \neq I_{V_1}$. Suppose $V_3 = \{y_j\}$, then $I_{V_3} = 2n + 1 \neq I_{V_1}$, a contradiction.

Suppose $V_r = \{x_{i1}, y_{i2}\}$. Then $I_{V_r} = 2n = a + 1 \Longrightarrow a = 2n - 1$. Suppose, without loss of generality, $V_2 = \{x_{i1}, x_{i2}, \cdots, x_{ir}\}, r \geqslant 2$, then $I_{V_2} = 2n - r + 1 = a + 1 \Longrightarrow a = 2n - r$. Therefore, $|V_1| + |V_2| = 2n$. Therefore, the EI-partition contains exactly one singleton set V_3 other than V_1 and V_2. If $V_3 = \{x_i\}$, then $I_{V_3} = 2n = I_{V_2} = 2n - r \Longrightarrow r = 0$, a contradiction.

If $V_3 = \{y_j\}$. then $I_{V_3} = 2n + 1 = I_{V_2} = 2n - r \Longrightarrow r = -1$, a contradiction.

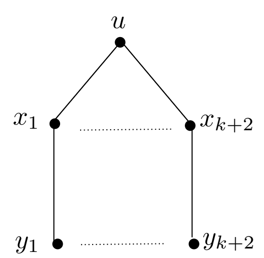
Therefore, $EI(H) = 3$ if there exist V_1, V_2 and V_3 with $|V_1| = 2n - 1, |V_2| = |V_3| = 1$ and each of V_2 and V_3 contains exactly one $x_i, 1 \leqslant i \leqslant n$. $EI(H) = 2$ if there exist V_1 and V_2 with $|V_1| = 2n - r + 1, |V_2| = r$ and each of V_1 and V_2 contains exactly one $y_j, 1 \leqslant j \leqslant n$.

Subcase (ii): $I_{V_1} = a + 2$.

Let $a = n + k (k \geqslant 1)$. $I_{V_1} = n - k + 1$. $V_2 \cup V_3$ contains $n - k + 1$ vertices. Suppose $V_3 \neq \emptyset$. Then, $\alpha_1 + \beta_1 + \beta_2 = |V_2| \leqslant n - k$. $I_{V_2} = 2n - \alpha_1 + \beta_1 + 1 = n + k + 2$. Therefore, $\alpha_1 = \beta_1 + n + k - 1, n \geqslant k$, if $\beta_1 \geqslant 1$. But $\alpha_1 \leqslant |V_2| \leqslant n - k$. Therefore, $\alpha_1 \leqslant n - k$. Thus, $\alpha_1 = n - k$ and hence $\beta_1 = 0$, a contradiction (since $\beta_1 \geqslant 1$).

Therefore, $\alpha_1 = n - k - 1$ and hence $\beta_1 = 0$, $I_{V_2} = n + k + 2$. V_2 contains $n - k - 1, x_i' s$. $|V_3| \leqslant 2$. If $|V_2| = n - k - 1$ then $|V_3| = 2$.

Suppose V_3 contains one x_i and one y_j. If $j = i$, then $I_{V_3} = 2n+1 = n+k+2 \Longrightarrow n = k + 1$. Therefore, $|V_2| = 0$, a contradiction. If $j \neq i$, then $I_{V_2} = 2n = n + k + 2 \Longrightarrow n = k + 2$. Therefore, $|V_2| = 1$. Thus, V_2 contains exactly one x_i , since $\alpha_1 = 1$.



Then $\{V_1 = \{u, y_2, \cdots, y_{k+2}, x_3, \cdots, x_{k+2}\}; V_2 = \{x_1\}; V_3 = \{x_2, u_1\}\}$ is a EI-partition of G.

Suppose V_3 contains two $y_j' s$. Then $I_{V_3} = 2n + 1 = n + k + 2 \Longrightarrow n = k + 1$, a contradiction, since $|V_2| = 0$. Suppose V_3 contains two $x_i' s$. Then $I_{V_3} = 2n - 1 = n + k + 2 \Longrightarrow n = k + 3$. Therefore, $|V_2| = 2$ and $\alpha_1 = 2$. Therefore, V_1 contains all $y_j' s$, a contradiction.

Suppose $V_4 \neq \emptyset$ and $|V_3 \cup V_4| = 2$. Then $|V_3| = 1 = |V_4|$. If V_3 and V_4 each contains exactly one x_i, then $I_{V_3} = I_{V_4} = 2n = n+k+2 \Longrightarrow n = k+2$. V_2 contains exactly one x_i , since $\alpha_1 = 1$. Therefore, all $y_j' s$ are contained in V_1, a contradiction. If V_3 contains one x_i and V_4 each contains one y_j , then $I_{V_2} \neq I_{V_4}$, a contradiction. If V_3 and V_4 each contains exactly y_j, then $I_{V_3} = 2n+1 = n+k+2 \Longrightarrow n = k+1$. Therefore, $|V_2| = 0$, a contradiction. Suppose $|V_2| = n-k$. Then $|V_3| = 1$. If V_3 contains exactly one y_j . Then $I_{V_2} = 2n+1 = n+k+2 \Longrightarrow n = k+1$. Therefore, $|V_2| = 1$ and $\alpha_1 = 0$. V_2 contains exactly one y_j. Therefore, V_1 contains all $x_1' s$, a contradiction. If V_3 contains exactly one x_i, then $I_{V_3} = 2n = n+k+2 \Longrightarrow n = k+2$. Therefore, $|V_2| = 2$ and $\alpha_1 = 1$. Then EI-partition of G is given by $\{V_1 = \{u, y_1, y_2, \cdots, y_{k+2}, x_3, \cdots, x_{k+2}\}; V_2 = \{x_1, y_2\}; V_3 = \{x_2\}\}$. $I_{V_1} = I_{V_2} = I_{V_3} = n+k+2$.

Case (IV): Let $a \leqslant n$ and $b \leqslant n$.

Let $a = n-k (k \geqslant 0)$. Then $I_{V_1} = n-k+2$. $V_2 \cup V_3$ contains $n-k+1$ vertices. Suppose $V_3 \neq \emptyset$. Since $b < n, |V_2| \leqslant n-1$. Therefore, $\alpha_1 + \beta_1 + \beta_2 = |V_2| \leqslant n-1$. $I_{V_2} = 2n - \alpha_1 + \beta_1 + 1 = n-k+2$. $\alpha_1 = n+k+1+\beta_1$. Since β_1 and k are non-negative and since $\alpha_1 \leqslant |V_2|+ \leqslant n-1$, we get that $k = \beta_1 = 0$. Therefore, $a = n$.

Let $a = n$. Then $I_{V_1} = n+2$, $V_2 \cup V_3$ contains $n+1$ vertices. Suppose $V_3 \neq \emptyset$. Since $b < n, |V_2| \leqslant n-1$ ad $|V_3| \leqslant n-1$. Therefore, $\alpha_1 + \beta_1 + \beta_2 = |V_2| \leqslant n-1$. $I_{V_1} = 2n - \alpha_1 \leqslant |V_2| \leqslant n-1$. Therefore, $\beta_1 + n - 1 \leqslant n-1$. Therefore, $\beta_1 = 0$ and $\alpha_1 = n-1$. Therefore, $|V_2| = n-1$. Therefore, V_2 contains $n-1, x_i' s$ and no y_j. Therefore, $|V_3| \leqslant 2$.

Suppose $|V_3| = 2$. If V_3 contains one x_i, then V_1 contains no x_j. If V_3 contains one x_i and corresponding y_j, then $I_{V_3} = 2n+1 = n+2 \Longrightarrow n = 1$, a contradiction. If V_3 contains one x_i and one y_j , $j \neq i$, then $I_{V_3} = 2n = n+2 \Longrightarrow n = 2$, a contradiction. If V_3 contains two $y_j' s$, then $I_{V_3} = 2n+1 = n+2 \Longrightarrow n = 1$, a contradiction. Suppose $|V_3| = 1$ and $|V_4| = 1$. If either V_3 or V_4 contains one x_i, then V_1 does not contain any x_i. $I_{V_3} = 2n = n+2 \Longrightarrow n = 2$, a contradiction. □

THEOREM 2.6. *Let $H = G \cup tK_1$. Then $EI(H) = t+1$.*

PROOF. Let $V(H) = \{v_1, v_2, \cdots, v_n, u_1, \cdots, u_t\}$ and $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let $V_1 = V(G), V_2 = \{u_1\}, V_3 = \{u_2\}, \cdots, V_{t+1} = \{u_t\}$. Then $I_{V_1} = n+1$ and $I_{V_j} = n+1$, for all $j, 2 \leqslant j \leqslant t+1$. Therefore, $EI(H) \geqslant t+1$. Suppose $EI(H) \geqslant t+2$. Then there exists V_1, V_2 in π (which is an EI-partition) such that V_1 and V_2 are proper subsets of $V(G)$. Then $I_{V_1} = |V_1| + m(H - V_1)$. Suppose $m(H - V_1) = 1$. Then $I_{V_1} = |V_1| + 1 < n+1 = I_{V_j}$, where $V_j = \{u_{j-1}\}, j \geqslant 2$, a contradiction. Suppose $m(H - V_1) \geqslant 2$. Therefore, $|V_1| = n - m(H - V_1)$. Then $I_{V_1} = n - m(H - V_1) + m(H - V_1) = n < n+1 = I_{V_j}$, where $V_j = \{u_{j-1}\}, j \geqslant 2$, a contradiction. Therefore $EI(H) \leqslant t+1$. Hence $EI(H) = t+1$. □

THEOREM 2.7. *If $G = \bigcup_{i=1}^{k} G_i$, where each G_i is connected and G is a vertex disjoint union of some same order graphs $G_i, 1 \leqslant i \leqslant k$, then $E(G) = |V(G)|$.*

PROOF. Since $G_1, G_2, \cdots, G_k$ all have the same order and are all connected, for each vertex u of G, $I_u = |V(G_i)| + 1$ is a constant. Therefore, $EI(G) = |V(G)|$. □

THEOREM 2.8. *Let* $G = \bigcup_{i=1}^{k} G_i$, *where each* G_i *is connected.*

Let $\max_{1 \leqslant i \leqslant k} |V(G_i)| = t$.

(a) Suppose there exists G_{i1}, G_{i2} *such that* $|V(G_{i1})| = |V(G_{i2})| = t$. *Then* $EI(G) = |V(G)|$.

(b)Suppose there exists a unique $G_i, 1 \leqslant i \leqslant t$ *such that* $|V(G_i)| = t$.

(i) if there exists G_j *such that* $|V(G_j)| = t - 1, 1 \leqslant j \leqslant k$, *then*

$$EI(G) = \begin{cases} |V(G)| - \frac{t}{2} & \text{if } t \text{ is even} \\ |V(G)| - \frac{t+1}{2} & \text{if } t \text{ is odd} \end{cases}$$

(ii) Suppose there exists no G_j *such that* $|V(G_j)| = t-1$. *Let* $\max_{|V(G_i)|<t} |V(G_i)| = t_1$, *where* $t_1 < t - 1.$, *Then* $EI(G) = n - t - \lfloor \frac{t}{t-t_1+1} \rfloor$.

PROOF. (a) In this case, for any vertex u of G, $I_u = t+1 =$ constant. Therefore, $EI(G) = |V(G)|$.

(b) (i) In this case, for any vertex u of $V(G_l), l \neq i, l \neq j$, $I_u = t + 1 = 1 =$ a constant. For any vertex $u \in V(G_i), I_u = t$. Consider $S = \{u, v\}$, where $u \in V(G_i)$ and $v \in V(G_j)$. Then, $I_s = t + 1$. Also, for any $u_1, u_2 \in V(G_i), I_{\{u_1,u_2\}} = t + 1$.

Therefore,

$$EI(G) = \begin{cases} |V(G)| - \frac{t}{2} & \text{if t is even} \\ |V(G)| - \frac{t+1}{2} & \text{if t is odd} \end{cases}$$

(ii) Let $t = \lambda(t - t_1 + 1) + \mu$, where $0 \leqslant \mu < t - t_1 + 1$. For any $t - t_1 + 1$ vertices of $V(G_i)$ constituting a set say S, $I_S = t - t_1 + 1 = t + 1$ (note that $m(G_i - S) \leqslant t - (t - t_1) + 1 = t_1 + 1$ and hence $|S| + m(G_i - S) \leqslant t - t_1 + 1 + t_1 = t < (t + 1)$. Then a set S_1 of at most μ vertices of $V(G_i)$ has $I_{S_1} = t + 1$. But $I_{S_1} = \mu + t < t - t_1 + 1 = t + 1$. That is, $I_{S_1} < t + 1$, a contradiction. Therefore, $EI(G) \leqslant n - t + \mu$. Therefore, $EI(G) \leqslant n - t + \lambda = n - t + \lfloor \frac{t}{t-t_1+1} \rfloor$. □

REMARK 2.3. Integrity is a vulnerability parameter and it gives a measure of the strength of the network to withstand the failure of certain nodes. If the network is capable of being divided into sub networks, each of which has the same integrity, then the failure in any sub network may be managed in the same way as in any other sub network and that in the event of an attack on the net work, it is possible to remedy it since all sub networks are of equal integrity.

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JRF, Ramanujan Research Center in Mathematics, Saraswathi Narayanan College, Madurai, India

E-mail address: neyamsundar@yahoo.com

Head, Ramanujan Research Center in Mathematics, Saraswathi Narayanan College, Madurai, India

E-mail address: sulanesri@yahoo.com