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Inequalities Involving Hyperbolic Functions and Trigonometric Functions

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Abstract. Inequalities involving hyperbolic functions and trigonometric functions are established. Most of the established inequalities can be regarded as the companion inequalities to those obtained by Lazarević, Adamović and Mitrinović and Cusa and Huygens The established results are obtained with the aid of the Schwab-Borchardt mean.

1. Introduction

In recent years the following two-sided trigonometric inequality for hyperbolic functions

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{\cosh x + 2}{3} \tag{1.1}$$

($x \neq 0$) has attracted attention of several researchers. The left inequality in (1.1) has been obtained by Lazarević (see, e.g. [**2**]). The counterpart of (1.1) for trigonometric functions

$$(\cos \varphi)^{1/3} < \frac{\sin \varphi}{\varphi} < \frac{\cos \varphi + 2}{3} \tag{1.2}$$

($0 < |\varphi| < \frac{\pi}{2}$) is also well-known. The left inequality in (1.2) have been proven by Adamović and Mitrinović (see[**6**]), while the second one is due to Cusa and Huygens (see [**17, 4**]). Inequalities mentioned above also have been obtained in [**13**]. Generalizations of inequalities (1.1) and (1.2) to Jacobian elliptic functions are established in [**8**].

For recent development in theory of inequalities for the hyperbolic and trigonometric functions the interested reader is referred to [**5**], [**14**], [**7**], [**9**], [**15**], [**16**] and the references therein.

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The goal of this paper is to derive inequalities involving hyperbolic and trigonometric functions. Most of them are the two - sided inequalities which are similar to inequalities (1.1) and (1.2). In Section 2 we recall definition and basic properties of the Schwab-Borchardt mean. Definitions of four particular bivariate means, which can be regarded as special cases of the Schwab-Borchardt mean, are also included in this section. The main results of this note are established in Section 3.

2. Definitions and Preliminaries

The geometric, arithmetic, and the root-mean square means of $a > 0$ and $b > 0$ will be denoted by G, A, and Q, respectively, and they are defined as follows

$$G \equiv G(a,b) = \sqrt{ab}, \quad A \equiv A(a,b) = \frac{a+b}{2}\,, \quad Q \equiv Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}\,.$$

Other bivariate means used in the subsequent sections include the logarithmic mean

$$L = \frac{u}{\tanh^{-1} u} A, \tag{2.1}$$

the first and second Seiffert means P and T, where

$$P = \frac{u}{\sin^{-1} u} A \tag{2.2}$$

and

$$T = \frac{u}{\tan^{-1} u} A \tag{2.3}$$

(see [**18**], [**19**], [**11**], [**9**]). Here

$$u = \frac{a-b}{a+b} \tag{2.4}$$

($a \neq b$). Another mean which is also of interest has been introduced in [**11**, (2.6)] and is defined as follows

$$M = \frac{u}{\sinh^{-1} u} A\,. \tag{2.5}$$

It is known that

$$G < L < P < A < M < T < Q$$

(see [**11**, (2.10)]). All the bivariate means mentioned above are strict, homogeneous of degree 1 and they are strictly increasing in each of its variables. Let the letter W stand for one of these means. Then homogenity of W implies that

$$W(a,b) = \sqrt{ab}W(e^x, e^{-x}),$$

where $x = \frac{1}{2}\ln(a/b)$.

Means L, P, T and M are special cases of the Schwab-Borchardt mean. For $a \geqslant 0$ and $b > 0$ this mean will be denoted by $SB(a,b) \equiv SB$. The Schwab-Borchardt mean is the iterative mean, i.e.,

$$SB = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n\,,$$

where

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_{n+1} b_n} \tag{2.6}$$

$(n = 0, 1, \ldots)$. It is known that

$$SB(a,b) = \begin{cases} \dfrac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)} & \text{if } a < b, \\ \dfrac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)} & \text{if } b < a \end{cases} \tag{2.7}$$

(see [**1**, Theorem 8.4], [**3**, (2.3)]). The mean SB is nonsymmetric, homogeneous of degree 1 and strictly increasing it its variables. It has been shown in [**11**] that

$$L = SB(A, G), \quad P = SB(G, A), \quad T = SB(A, Q), \quad M = SB(Q, A). \tag{2.8}$$

We close this section with some inequalities involving the Schwab - Borchardt mean. The following two-sided inequality

$$(ab^2)^{1/3} < SB(a,b) < \frac{a + 2b}{3} \tag{2.9}$$

is known (see [**11**]). Using the invariance property $SB(a_n, b_n) = SB(a, b)$, where $\{a_n\}$ and $\{b_n\}$ $(n = 0, 1, \ldots)$ are defined in (2.6), the last inequality can be generalized to

$$(a_n b_n^2)^{1/3} < SB(a,b) < \frac{a_n + 2b_n}{3} \tag{2.10}$$

(see [**11**]). It is worth mentioning that the sequence $\{(a_n b_n^2)^{1/3}\}_{n=0}^{\infty}$ is strictly increasing while $\{(a_n + 2b_n)/3\}_{n=0}^{\infty}$ is strictly decreasing provided $a \neq b$. For more inequalities for the Schwab - Borchardt mean the interested reader is referred to [**10**] and [**12**].

3. Main results

The goal of this section is to establish several inequalities for families of functions under discussion. We begin with inequalities for the hyperbolic functions.

THEOREM 3.1. *Let $x \neq 0$. Then*

$$(\cosh x)^{2/3} < \frac{\sinh x}{sin^{-1}(\tanh x)} < \frac{1 + 2\cosh x}{3}, \tag{3.1}$$

$$((\cosh 2x)^{1/2} \cosh^2 x)^{1/3} < \frac{\sinh x}{\sinh^{-1}(\tanh x)} < \frac{(\cosh 2x)^{1/2} + 2\cosh x}{3}, \tag{3.2}$$

and

$$((\cosh 2x) \cosh x)^{1/3} < \frac{\sinh x}{\tan^{-1}(\tanh x)} < \frac{2(\cosh 2x)^{1/2} + \cosh x}{3}. \tag{3.3}$$

PROOF. Let $(a,b)=(e^x,e^{-x})$. Then $G=1$, $A=\cosh x$, $Q=(\cosh 2x)^{1/2}$ and, using (2.4), $u=\tanh x$. Moreover, making use of (2.2), (2.5), and (2.3) we obtain

$$P=\frac{\sinh x}{\sin^{-1}(\tanh x)}, M=\frac{\sinh x}{\sinh^{-1}(\tanh x)}, T=\frac{\sinh x}{\tan^{-1}(\tanh x)}. \tag{3.4}$$

For the proof of (3.1) we utilize (2.9) with $a=G$ and $b=A$ followed by application of the second part of (2.8) and the first formula of (3.4) to obtain the desired result. The two-sided inequalities (3.2) and (3.3) can be established in a similar fashion. For instance, for the proof of (3.2) we use (2.9) with $a=Q$ and $b=A$ followed by use of the fourth formula of (2.8). Application the second formula of (3.4) yields the assertion. Finally, in order to prove (3.3) we follow the lines introduced above with $a=A$ and $b=Q$. □

Inequalities (1.1) can be established using the method utilized in the proof of Theorem 3.1. Letting $a=A$ and $b=G$ we obtain, using (2.8) and (2.1), $SB=L=\sinh x/x$. Application of (2.9) yields the desired result. It is worth mentioning that tighter inequalities than (1.1) and (3.1)- (3.3) can be obtained using (2.10) instead of (2.9). Unfortunately the resulting inequalities are more complicated than those included here. We omit further details.

Inequalities for trigonometric functions can be easily derived from (1.1) and from inequalities obtained in Theorem 3.1. To this aim it suffices to use a new variable φ which is defined as follows

$$\tanh x=\sin\varphi. \tag{3.5}$$

This implies that

$$\sinh x=\tan\varphi, \cosh x=\sec\varphi, x=\tanh^{-1}(\sin\varphi). \tag{3.6}$$

One can easily verify that (1.2) follows from (3.1) using (3.5) and (3.6). Application of (3.5) and (3.6) to (1.1) and to (3.2) - (3.3) yields

COROLLARY 3.1. *Let* $0<|\varphi|<\frac{\pi}{2}$. *Then*

$$(\cos\varphi)^{2/3}<\frac{\sin\varphi}{\tanh^{-1}(\sin\varphi)}<\frac{1+2\cos\varphi}{3}, \tag{3.7}$$

$$(1+\sin^2\varphi)^{1/6}<\frac{\sin\varphi}{\sinh^{-1}(\sin\varphi)}<\frac{2+(1+\sin^2\varphi)^{1/2}}{3}, \tag{3.8}$$

and

$$(1+\sin^2\varphi)^{1/3}<\frac{\sin\varphi}{\tan^{-1}(\sin\varphi)}<\frac{1+2(1+\sin^2\varphi)^{1/2}}{3}. \tag{3.9}$$

Let {(1.2), (3.7), (3.8), (3.9)} be a quadruple of two-sided inequalities established in this section. It is natural to ask whether members of this set are comparable, i.e., whether it is true that, in symbolic notation, the following chain of inequalities

$$(3.7)<(1.2)<(3.8)<(3.9) \tag{3.10}$$

is valid for all arguments φ which satisfy $0 < |\varphi| < \frac{\pi}{2}$. We have the following.

PROPOSITION 3.1. *First inequality in* (3.10) *is satisfied if* $0 < |\varphi| < 1.52$ *while the remaining ones are valid provided* $0 < |\varphi| < \frac{\pi}{2}$.

PROOF. We shall show that (3.7) < (1.2) holds for φ in the stated domain. To this aim it suffices to show that

$$\frac{1 + 2\cos\varphi}{3} < (\cos\varphi)^{1/3}.$$

Letting $\cos\varphi = a^3$ we see that the last inequality can be written as $2a^3 - 3a + 1 < 0$. Using MATLAB we see that the positive roots of the last cubic polynomial are equal to 0.366... and 1. Thus the last inequality is satisfied if $0.366 < a < 1$ or what is the same that $cos^{-1}(1) < |\varphi| < cos^{-1}(0.366^3)$. Proofs of the second and third inequalities in (3.10) are straightforward hence they are omitted. □

The last result of this section reads as follows.

THEOREM 3.2. *The following inequalities*

$$1 < \frac{\sinh x}{\sin^{-1}(\tanh x)} \frac{\tanh x}{x} \tag{3.11}$$

($x \neq 0$*) and*

$$1 < \frac{\sin\varphi}{\tanh^{-1}(\sin\varphi)} \frac{\tan\varphi}{\varphi} \tag{3.12}$$

($0 < |\varphi| < \frac{\pi}{2}$*) hold true.*

PROOF. First we use the left inequality in (2.9) with $a = A$ and $b = G$ followed by application of the first formula in (2.8) to obtain

$$(AG^2)^{1/3} < L.$$

Repeating these calculations now with $a = G$ and $b = A$ yields

$$(GA^2)^{1/3} < P.$$

Multiplying out the corresponding sides of the last two inequalities we obtain

$$AG < PL. \tag{3.13}$$

To obtain inequality (3.11) we let $(a, b) = (e^x, e^{-x})$ and utilize formulas for means A, G, L and P which appear in the proof of Theorem 3.1. This in conjunction with (3.13) gives the desired result. Inequality (3.12) follows from inequality (3.11) by using transformations (3.5) and (3.6). □

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