

One-step iterative scheme for approximating common fixed points of three multivalued nonexpansive mappings

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ABSTRACT. In this paper, we introduce a new one-step iterative scheme to approximate a common fixed point of three multivalued nonexpansive mappings in a uniformly convex real Banach space and establish strong and weak convergence theorems for the proposed process. Our results extend important results.

1. Introduction

Let E be a real Banach space. A subset K is called proximal if for each $x \in E$, there exists an element $k \in K$ such that

$$d(x, k) = \inf\{\|x - y\| : y \in K\} = d(x, K).$$

It is known that a weakly compact convex subsets of a Banach space and closed convex subsets of a uniformly convex Banach space are proximal. We shall denote the family of nonempty bounded proximal subsets of K by $P(K)$ and the family of nonempty compact proximal subsets of K by $CP(K)$. Consistent with [8], let $CB(X)$ be the class of all nonempty bounded and closed subsets of X . Let H be a Hausdorff metric induced by the metric d of X , given by

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\},$$

for every $A, B \in CB(X)$. It is obvious that $P(K) \in CB(X)$.

A multivalued mapping $T : K \rightarrow P(K)$ is said to be a contraction if there exists a constant $k \in [0, 1)$ such that for any $x, y \in K$,

$$H(Tx, Ty) \leq k\|x - y\|,$$

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and T is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|,$$

for all $x, y \in K$. A point $x \in K$ is called a fixed point of T if $x \in Tx$. Throughout the paper \mathbb{N} denotes the set of all positive integers and the $F(T)$ denotes the set of all fixed point of T .

The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [7] (see also [8]). Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics. The theory of multivalued nonexpansive mappings is harder than the corresponding theory of single valued nonexpansive mappings. Different iterative processes have been used to approximate the fixed points of multivalued nonexpansive mappings. Among these iterative processes, Sastry and Babu [12] considered the following:

Let $T : X \rightarrow P(X)$ be a mapping with $p \in Tp$.

(I) Define a sequence by $x = x_1 \in X$,

$$x_{n+1} = (1 - a_n)x_n + a_n y_n,$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$, and $\{a_n\}$ is a sequence in $(0, 1)$ satisfying $\lim_{n \rightarrow \infty} a_n = 0$ and $\sum a_n = \infty$.

(II) The sequence of Ishikawa iterates is defined by

$$\begin{aligned} x_1 &= x \in X, \\ y_n &= (1 - b_n)x_n + b_n z_n \\ x_{n+1} &= (1 - a_n)x_n + a_n u_n, \end{aligned}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ are such that $\|z_n - p\| = d(p, Tx_n)$ and $\|u_n - p\| = d(p, Ty_n)$, and $\{a_n\}, \{b_n\}$ are real sequences of numbers with $0 \leq a_n, b_n < 1$ satisfying $\lim_{n \rightarrow \infty} b_n = 0$ and $\sum a_n b_n = \infty$.

Panyanak [11] obtained the following theorem.

THEOREM 1.1. *Let K be a nonempty compact convex subset of a uniformly convex Banach space X . Suppose that a nonexpansive map $T : K \rightarrow P(K)$ has a fixed point p . Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by (II). Then $\{x_n\}$ converges to a fixed point of T .*

Following is the useful Lemma due to Nadler [8].

LEMMA 1.1. *Let $A, B \in CB(X)$ and $a \in A$. If $\eta > 0$, then exists $b \in B$ such that $d(a, b) \leq H(A, B) + \eta$*

Based on above lemma, Song and Wang [13] modified the iteration scheme used in [11] and improved the results presented therein. This scheme reads as follows:

The sequence of Ishikawa iterates is defined by:

$$\begin{aligned} x_1 &= x \in X, \\ y_n &= (1 - b_n)x_n + b_n z_n \\ x_{n+1} &= (1 - a_n)x_n + a_n u_n, \end{aligned}$$

where $z_n \in Tx_n, u_n \in Ty_n$ are such that $\|z_n - u_n\| \leq H(Tx_n, Ty_n) + \eta_n$ and $\|z_{n+1} - u_n\| \leq H(Tx_{n+1}, Ty_n) + \eta_n$, and $\{a_n\}, \{b_n\}$ are real sequences of numbers with $0 \leq a_n, b_n < 1$ satisfying $\lim_{n \rightarrow \infty} b_n = 0$ and $\sum a_n b_n = \infty$.

On the other hand, T. Hu et al. [3] obtained common fixed point of two multivalued mappings $T, S : K \rightarrow CB(K)$ satisfying certain contractive condition. To achieve this, they employed the following iterative process:

$$\begin{aligned} x_1 &= x \in K, \\ y_n &= (1 - b_n)x_n + b_n v_n, \quad v_n \in Tx_n \\ x_{n+1} &= (1 - a_n)x_n + a_n u_n, \quad u_n \in Sy_n, \quad n \in \mathbb{N}, \end{aligned}$$

where $\{a_n\}$ and $\{b_n\}$ are in $[0, 1]$ satisfying certain conditions.

Recently, S. H. Khan et al.[4] introduces new one-step iterative process to compute the common fixed points of two multivalued nonexpansive mappings. Let $S, T : K \rightarrow CB(K)$ be two multivalued nonexpansive mappings with a common fixed point p . They employed the following iterative process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= \lambda y_n + (1 - \lambda)z_n, \quad n \in \mathbb{N}, \end{aligned}$$

where $y_n \in Sx_n$ such that $\|y_n - y_{n+1}\| \leq H(Sx_n, Sx_{n+1}) + \eta_n, z_n \in Tx_n$ such that $\|z_n - z_{n+1}\| \leq H(Tx_n, Tx_{n+1}) + \eta_n$ and λ is a real number satisfying $0 < \lambda < 1$. $S, T : K \rightarrow CB(K)$ are said to satisfy condition (C) if $d(x, y) \leq d(z, y)$, for $y \in Sx$ and $z \in Tx$.

Very recently, M. Abbas et al.[1] introduce a new one-step iterative process to compute common fixed points of two multivalued nonexpansive mappings. Let $S, T : K \rightarrow CB(K)$ be two multivalued nonexpansive mappings. They employed the following iterative process:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= a_n x_n + b_n y_n + c_n z_n, \quad n \in \mathbb{N}, \end{aligned}$$

where $y_n \in Tx_n$ and $z_n \in Sx_n$ such that $\|y_n - p\| \leq d(p, Sx_n)$ and $\|z_n - p\| \leq d(p, Tx_n)$, whenever p is fixed point of any one of the mappings S and T , and $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequence of numbers in $(0, 1)$ satisfying $a_n + b_n + c_n \leq 1$.

In this paper, we introduce new one-step iterative process to approximate the common fixed points of three multivalued nonexpansive mappings in a uniformly convex real Banach space and establish strong and weak convergence theorems for the proposed process. Our results extend and improve the recent results in [1, 4].

Let $T, S, R : K \rightarrow CB(K)$ be three multivalued nonexpansive mappings with common fixed point p . Our process reads as follows:

$$(1.1) \quad \begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= \alpha_n y_n + \beta_n z_n + \gamma_n w_n \quad n \in N, \end{aligned}$$

where $y_n \in Tx_n$ such that $\|y_n - y_{n+1}\| \leq H(Tx_n, Tx_{n+1}) + \eta_n$, $z_n \in Sx_n$ such that $\|z_n - z_{n+1}\| \leq H(Sx_n, Sx_{n+1}) + \eta_n$, $w_n \in Rx_n$ such that $\|w_n - w_{n+1}\| \leq H(Rx_n, Rx_{n+1}) + \eta_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence of numbers in $(0, 1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$. $T, S, R : K \rightarrow CB(K)$ are said to satisfy condition (C') if $d(x, y) \leq d(z, y)$, for $z \in Tx$ and $y \in Sx$ or $d(x, y) \leq d(z, y)$, for $z \in Tx$ and $y \in Rx$, or $d(x, y) \leq d(z, y)$, for $z \in Rx$ and $y \in Sx$.

We prove that the iterative process defined by (1.1) converges weakly under condition (C') and strongly under the condition (A') (to be defined later) to a common fixed point.

The following is an example of three multivalued nonexpansive mappings with a common fixed point satisfying condition (C') .

EXAMPLE 1.1. Let $X = [0, 1]$. Define $T, S, R : X \rightarrow CB(X)$ as follows:

$$Tx = \left[0, \frac{2x+1}{4}\right],$$

$$Sx = \left[0, \frac{4x+1}{6}\right],$$

and

$$Rx = \{x\}.$$

Then clearly T, S and R satisfy condition (C') , and have a common fixed point at $\frac{1}{2}$.

Let us recall the following definitions.

DEFINITION 1.1. [10] A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E , $x_n \rightharpoonup x$ (\rightharpoonup denotes weak convergence) implies that $\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying this condition are Hilbert spaces and all l^p .

Khan and Fukhar-ud-din [5] introduced the so-called condition (\dot{A}) and gave a bit improved version in [2]. The following is the multivalued version of condition (\dot{A}) .

DEFINITION 1.2. Three multivalued nonexpansive mappings $T, S, R : K \rightarrow CB(K)$ where K a subset of E , are said to satisfy condition (\dot{A}) if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that either $d(x, Tx) \geq f(d(x, F))$ or $d(x, Sx) \geq f(d(x, F))$ or $d(x, Rx) \geq f(d(x, F))$ for all $x \in K$.

DEFINITION 1.3. A multivalued mapping $T : K \rightarrow P(X)$ is called demiclosed at $y \in K$ if for any sequence $\{x_n\}$ in K which is weakly convergent to an element x and $y_n \in Tx_n$ with $\{y_n\}$ converges strongly to y , we have $y \in Tx$.

It is known that if K is a nonempty weakly compact convex subset of Banach space E satisfying Opial's condition and T a multivalued nonexpansive mapping of K into $K(X)$, then $I - T$ is demiclosed with respect to zero.

Next we state the following useful lemma.

LEMMA 1.2. [9] Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequence in uniformly convex Banach space X . Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\limsup_{n \rightarrow \infty} \|x_n\| = d$, $\limsup_{n \rightarrow \infty} \|y_n\| = d$, $\limsup_{n \rightarrow \infty} \|z_n\| = d$ and $\lim_{n \rightarrow \infty} \|\alpha_n x_n + \beta_n y_n + \gamma_n z_n\| = d$. If $\liminf_{n \rightarrow \infty} \alpha_n > 0$, $\liminf_{n \rightarrow \infty} \beta_n > 0$ and $\liminf_{n \rightarrow \infty} \gamma_n > 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = \lim_{n \rightarrow \infty} \|x_n - z_n\| = \lim_{n \rightarrow \infty} \|y_n - z_n\| = 0$.

2. Main Results

In this section, we shall prove the following lemmas. In the sequel, we will write $F = F(T) \cap F(S) \cap F(R)$ for the set of all common fixed points of the mappings T, S and R .

LEMMA 2.1. Let E be a normed space and K a nonempty closed convex subset. Let $T, S, R : K \rightarrow CB(K)$ be multivalued nonexpansive mappings. Let $\{x_n\}$ be the sequence as defined in (1.1). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$ then $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for all $p \in F$.

PROOF. Assume that $F \neq \emptyset$. Let $p \in F$. Then

$$\begin{aligned}
 \|x_{n+1} - p\| &= \|\alpha_n y_n + \beta_n z_n + \gamma_n w_n - p\| \\
 &= \|\alpha_n (y_n - p) + \beta_n (z_n - p) + \gamma_n (w_n - p)\| \\
 &\leq \alpha_n \|y_n - p\| + \beta_n \|z_n - p\| + \gamma_n \|w_n - p\| \\
 &\leq \alpha_n d(y_n, Tp) + \beta_n d(z_n, Sp) + \gamma_n d(w_n, Rp) \\
 &\leq \alpha_n H(Tx_n, Tp) + \beta_n H(Sx_n, Sp) + \gamma_n H(Rx_n, Rp) \\
 &\leq \alpha_n \|x_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| \\
 (2.1) \qquad &= \|x_n - p\|.
 \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for each $p \in F$.

LEMMA 2.2. Let E be a uniformly convex Banach space and K be nonempty closed convex subset. Let $T, S, R : K \rightarrow CB(K)$ be multivalued nonexpansive mappings satisfying condition (C') and $\{x_n\}$ be the sequence as defined in (1.1). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$ then $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = \lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0 = \lim_{n \rightarrow \infty} d(x_n, Rx_n)$.

PROOF. By Lemma (2.1), $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. We suppose that $\lim_{n \rightarrow \infty} \|x_n - p\| = c$ for some $c \geq 0$. Since T, S, R are nonexpansive mappings and $F \neq \emptyset$, we

have $\|y_n - p\| = d(y_n, Sp) \leq H(Tx_n, Tp) \leq \|x_n - p\|$ for each $p \in F$. Taking \limsup on both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|y_n - p\| \leq c.$$

Similarly,

$$\limsup_{n \rightarrow \infty} \|z_n - p\| \leq c,$$

and ,

$$\limsup_{n \rightarrow \infty} \|w_n - p\| \leq c.$$

Further,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = c,$$

implies that

$$\lim_{n \rightarrow \infty} \|\alpha_n(y_n - p) + \beta_n(z_n - p) + \gamma_n(w_n - p)\| = c.$$

Applying lemma (1.2), we get

$$\lim_{n \rightarrow \infty} \|y_n - z_n\| = 0.$$

But from the condition (C') we obtain that $d(x_n, y_n) \leq d(z_n, y_n)$,

$$\limsup_{n \rightarrow \infty} d(x_n, y_n) \leq 0.$$

That is,

$$(2.2) \quad \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

Also from lemma (1.2) and (2.2) we obtain

$$\|x_n - z_n\| \leq \|x_n - y_n\| + \|y_n - z_n\|,$$

implies that

$$(2.3) \quad \lim_{n \rightarrow \infty} \|x_n - z_n\| = 0.$$

Also from lemma (1.2) and (2.3) we have

$$\|x_n - w_n\| \leq \|x_n - z_n\| + \|z_n - w_n\|,$$

implies that

$$(2.4) \quad \lim_{n \rightarrow \infty} \|x_n - w_n\| = 0.$$

Now

$$d(x_n, Tx_n) \leq d(x_n, y_n),$$

also

$$d(x_n, Sx_n) \leq d(x_n, z_n),$$

and

$$d(x_n, Rx_n) \leq d(x_n, w_n),$$

gives $d(x_n, Tx_n) \rightarrow 0$, $d(x_n, Sx_n) \rightarrow 0$ and $d(x_n, Rx_n) \rightarrow 0$ as $n \rightarrow \infty$.

We now approximate common fixed points of the mappings T , S and R through weak convergence of the sequence $\{x_n\}$ defined in (1.1) as follows:

THEOREM 2.1. *Let E be a uniformly convex Banach space satisfying the Opial's condition. Let K be a nonempty closed convex subset of E and $T, S, R : K \rightarrow CB(K)$. If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, $I - T$, $I - S$ and $I - R$ are demiclosed with respect to zero, then $\{x_n\}$ converges weakly to a common fixed point of T , S and R .*

PROOF. Let $p \in F$. Then as proved in lemma (2.1), $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F . To prove this, let z_1, z_2 and z_3 be weak limits of the subsequences $\{x_{n_i}\}$, $\{x_{n_j}\}$ and $\{x_{n_l}\}$ of $\{x_n\}$, respectively. By lemma (2.2), there exists $y_n \in Tx_n$ such that $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ and $I - T$ is demiclosed with respect to zero, therefore we obtain $z_1 \in Tz_1$. Similarly, $z_2 \in Sz_2$ and $z_3 \in Rz_3$. Again in the same way, we can prove that $z_1, z_2, z_3 \in F$. Next, we prove uniqueness. For this suppose that $z_1 \neq z_2 \neq z_3$. Then by the Opial's condition,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - z_1\| &= \lim_{n_i \rightarrow \infty} \|x_{n_i} - z_1\| \\ &< \lim_{n_i \rightarrow \infty} \|x_{n_i} - z_2\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_2\| \\ &= \lim_{n_j \rightarrow \infty} \|x_{n_j} - z_2\| \\ &< \lim_{n_j \rightarrow \infty} \|x_{n_j} - z_3\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_3\| \\ &= \lim_{n_l \rightarrow \infty} \|x_{n_l} - z_3\| \\ &< \lim_{n_l \rightarrow \infty} \|x_{n_l} - z_1\| \\ &= \lim_{n \rightarrow \infty} \|x_n - z_1\|. \end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in F .

REMARK 2.1. Let C be a nonempty compact convex subset of a uniformly convex Banach space E satisfying Opial's condition, $T, S, R : C \rightarrow K(C)$ be multivalued nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (1.1), where $K(C)$ is the family of nonempty compact subsets of C . If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$ then $\{x_n\}$ converges weakly to a common fixed point of T , S and R .

The following result gives a necessary and sufficient condition for strong convergence of the sequence in (1.1) to a common fixed point of three mappings on a real Banach space.

THEOREM 2.2. *Let E be a real Banach space and $K, \{x_n\}, T, S, R$ be as in Lemma(2.2). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, then $\{x_n\}$ converges strongly to a common fixed point of T, S and R if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.*

PROOF. The necessity is obvious. Conversely, suppose that $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. As proved in lemma(2.1),

$$\|x_{n+1} - p\| \leq \|x_n - p\|.$$

This gives

$$d(x_{n+1}, F) \leq d(x_n, F),$$

so that $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. But, by hypothesis, $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. Therefore we must have $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence in K . Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, there exists a constant n_0 such that for all $n \geq n_0$, we have

$$\lim_{n \rightarrow \infty} d(x_n, F) < \frac{\varepsilon}{4}.$$

In particular, $\inf\{\|x_{n_0} - p\| : p \in F\} < \frac{\varepsilon}{4}$. There must exist a $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\varepsilon}{2}.$$

Now for $m, n \geq n_0$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\ &\leq 2\|x_{n_0} - p^*\| \\ &< 2\left(\frac{\varepsilon}{2}\right) = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E , and therefore it must converge in K . Let $\lim_{n \rightarrow \infty} x_n = q$. Now

$$\begin{aligned} d(q, Tq) &\leq d(q, x_n) + d(x_n, Tx_n) + H(Tx_n, Tq) \\ &\leq d(q, x_n) + d(x_n, y_n) + d(x_n, q) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

gives that $d(q, Tq) = 0$ which implies that $q \in Tq$. Similarly,

$$\begin{aligned} d(q, Sq) &\leq d(q, x_n) + d(x_n, Sx_n) + H(Sx_n, Sq) \\ &\leq d(q, x_n) + d(x_n, z_n) + d(x_n, q) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

gives that $d(q, Sq) = 0$ which implies that $q \in Sq$. Similarly,

$$\begin{aligned} d(q, Rq) &\leq d(q, x_n) + d(x_n, Rx_n) + H(Rx_n, Rq) \\ &\leq d(q, x_n) + d(x_n, z_n) + d(x_n, q) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

implies that $q \in Rq$. Consequently, $q \in F$.

We shall use the condition (A) to study the strong convergence of $\{x_n\}$ defined in (1.1). In our next theorem, we assume that $T, S, R : K \rightarrow CB(K)$ satisfy condition (A).

THEOREM 2.3. *Let E be a uniformly convex Banach space and $K, \{x_n\}$ be as in Lemma (2.2). Let $T, S, R : K \rightarrow CB(K)$ be three multivalued nonexpansive mappings satisfying condition (A). If $F \neq \emptyset$ and $Tp = Sp = Rp = \{p\}$ for any $p \in F$, then $\{x_n\}$ converges strongly to a common fixed point of T, S and R .*

PROOF. By lemma (2.2), $\lim_{n \rightarrow \infty} \|x_n - F\|$ exists for all $p \in F$. Call it c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. Now $\|x_{n+1} - p\| \leq \|x_n - p\|$ gives $\inf_{p \in F} \|x_{n+1} - p\| \leq \inf_{p \in F} \|x_n - p\|$, which implies that $d(x_{n+1}, F) \leq d(x_n, F)$ and so $\lim_{n \rightarrow \infty} d(x_n, F)$ exists. By using condition (A) either

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0,$$

or

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0,$$

or

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) \leq \lim_{n \rightarrow \infty} d(x_n, Rx_n) = 0.$$

In both the cases,

$$\lim_{n \rightarrow \infty} f(d(x_n, F)) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. An application of theorem (2.2) now gives the desired result.

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