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Eigenvalues for Iterative Systems of Nonlinear Second Order Boundary Value Problems on Time Scales

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ABSTRACT. Values of $\lambda_1, \lambda_2, ..., \lambda_n$ are determined for which there exist positive solutions of the iterative system of dynamic equations,

$$u_i^{\Delta\Delta}(t) + \lambda_i a_i(t) f_i(u_{i+1}(\sigma(t))) = 0,\ 1 \leqslant i \leqslant n,\ u_{n+1}(t) = u_1(t),$$

for $t \in [a,b]_{\mathbb{T}}$, and satisfying the boundary conditions, $u_i(a) = 0 = u_i(\sigma^2(b))$, $1 \leqslant i \leqslant n$, where $\mathbb{T}$ is a time scale. A Guo-Krasnosel'skii fixed-point theorem is applied.

1. Introduction

Let $\mathbb{T}$ be a time scale with $a, \sigma^2(b) \in \mathbb{T}$. Given an interval J of $\mathbb{R}$, we will use the interval notation,

$$J_{\mathbb{T}} = J \cap \mathbb{T}.$$

We are concerned with determining values of λ_i, $1 \leqslant i \leqslant n$, for which there exist positive solutions for the iterative system of dynamic equations,

$$\begin{aligned} u_i^{\Delta\Delta}(t) + \lambda_i a_i(t) f_i(u_{i+1}(\sigma(t))) = 0,\ 1 \leqslant i \leqslant n,\ t \in [a,b]_{\mathbb{T}}, \\ u_{n+1}(t) = u_1(t),\ t \in [a,b]_{\mathbb{T}}, \end{aligned} \tag{1.1}$$

satisfying the boundary conditions,

$$u_i(a) = 0 = u_i(\sigma^2(b)),\ 1 \leqslant i \leqslant n, \tag{1.2}$$

where

(A1) $f_i \in C([0,\infty),[0,\infty)),\ 1 \leqslant i \leqslant n$;
(A2) $a_i \in C([a,\sigma(b)]_{\mathbb{T}},[0,\infty)),\ 1 \leqslant i \leqslant n$, and each does not vanish identically on any closed subinterval of $[a,\sigma(b)]_{\mathbb{T}}$;

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(A3) Each of $f_{i0} := \lim_{x\to 0^+} \frac{f_i(x)}{x}$, $f_{i\infty} := \lim_{x\to\infty} \frac{f_i(x)}{x}$, $1 \leqslant i \leqslant n$, exists as positive real number.

For several years now, there has been a great deal of activity in studying on positive solutions of boundary value problems for ordinary differential equations. Interest in such solutions is high from a theoretical sense [**10, 12, 15, 18, 25**] and as applications for which only positive solutions are meaningful [**2, 11, 19, 20**]. These considerations are caste primarily for scalar problems, but good attention has been given to boundary value problems for systems of differential equations [**14, 15, 16, 17, 22, 24, 26**].

There is a great deal of research activity devoted to positive solutions of dynamic equations on time scales; see for example [**1, 3, 4, 5, 7, 9, 13**]. This work entails an extension of the paper by Chyan and Henderson [**9**] to eigenvalue problems for systems of nonlinear boundary value problems on time scales, and also, in a very real sense, an extension of recent paper by Benchohra, Henderson and Ntouyas [**6**]. Also, in that light, this paper is closely related to the works by Li and sun [**21, 23**].

The main tool in this paper is an application of the Guo-Krasnoselskii fixed point theorem for operators leaving a Banach space cone invariant [**12**]. A Green function plays a fundamental role in defining an appropriate operator on a suitable cone.

2. Green's Function and Bounds

In this section, we state the well-known Guo-Krasnosel'skii fixed point-theorem which we will apply to a completely continuous operator whose kernel, $G(t,s)$, is the Green's function for

$$-y^{\Delta\Delta} = 0, \tag{2.1}$$

$$y(a) = 0,\ y(\sigma^2(b)) = 0 \tag{2.2}$$

is given by

$$G(t,s) = \begin{cases} \frac{(t-a)(\sigma^2(b)-\sigma(s))}{\sigma^2(b)-a} : & a \leqslant t \leqslant s \leqslant \sigma^2(b) \\ \frac{(\sigma(s)-a)(\sigma^2(b)-t)}{\sigma^2(b)-a} : & a \leqslant \sigma(s) \leqslant t \leqslant \sigma^2(b). \end{cases} \tag{2.3}$$

One can easily check that

$$G(t,s) > 0,\ (t,s) \in (a, \sigma^2(b))_{\mathbb{T}} \times (a, \sigma(b))_{\mathbb{T}} \tag{2.4}$$

and

$$G(t,s) \leqslant G(\sigma(s), s) = \frac{(\sigma^2(b)-\sigma(s))(\sigma(s)-a)}{\sigma^2(b)-a} \tag{2.5}$$

for $t \in [a, \sigma^2(b)]_{\mathbb{T}}$, $s \in [a, \sigma(b)]_{\mathbb{T}}$, and let $I = [\frac{3a+\sigma^2(b)}{4}, \frac{a+3\sigma^2(b)}{4}]_{\mathbb{T}}$ then

$$G(t,s) \geqslant kG(\sigma(s), s) = k\frac{(\sigma(s)-a)(\sigma^2(b)-\sigma(s))}{\sigma^2(b)-a} \tag{2.6}$$

for $t \in I$, $s \in [a, \sigma(b)]_{\mathbb{T}}$, where

$$k = \min\left\{\frac{1}{4}, \frac{\sigma^2(b) - a}{4(\sigma^2(b) - \sigma(a))}\right\}.$$

We note that an n-tuple $(u_1(t), u_2(t), ..., u_n(t))$ is a solution of the eigenvalue problem (1.1)-(1.2) if and only if

$$u_i(t) = \lambda_i \int_a^{\sigma(b)} G(t,s)a_i(s)f_i(u_{i+1}(\sigma(s)))\Delta s,\ a \leqslant t \leqslant \sigma^2(b),\ 1 \leqslant i \leqslant n,$$

and

$$u_{n+1}(t) = u_1(t),\ a \leqslant t \leqslant \sigma^2(b),$$

so that, in particular,

$$\begin{aligned} u_1(t) = \lambda_1 \int_a^{\sigma(b)} & G(t,s_1)a_1(s_1)f_1\Big(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1),s_2)a_2(s_2)\times \\ & \times f_2\Big(\lambda_3 \int_a^{\sigma(b)} G(\sigma(s_2),s_3)a_3(s_3)...\times \\ & \times f_{n-1}\Big(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u_1(\sigma(s_n)))\Delta s_n\Big)...\Delta s_3\Big)\Delta s_2\Big)\Delta s_1. \end{aligned}$$

Values of $\lambda_1, \lambda_2, ..., \lambda_n$ for which there are positive solutions (positive with respect to a cone) of (1.1)-(1.2), will be determined via applications of the following fixed-point theorem [**12**].

THEOREM 2.1. (***Krasnosel'skii***) *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that either

(i) $\|Tu\| \leqslant \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \geqslant \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2,$ *or*
(ii) $\|Tu\| \geqslant \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_1,$ *and* $\|Tu\| \leqslant \|u\|,\ u \in \mathcal{P} \cap \partial\Omega_2.$

Then T has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Positive Solutions in a Cone

In this section, we apply Theorem 2.1 to obtain solutions in a cone (i.e., positive solutions) of (1.1)-(1.2). Assume throughout that $[a, \sigma^2(b)]_{\mathbb{T}}$ is such that

$$\xi = \min\left\{t \in \mathbb{T} : t \geqslant \frac{3a + \sigma^2(b)}{4}\right\},$$

and

$$\omega = \max\left\{t \in \mathbb{T} : t \leqslant \frac{a + 3\sigma^2(b)}{4}\right\}$$

both exist and satisfy

$$\frac{3a + \sigma^2(b)}{4} \leqslant \xi < \omega \leqslant \frac{a + 3\sigma^2(b)}{4}.$$

Next, let $\tau_i \in [\xi, \omega]_{\mathbb{T}}$ be defied by

$$\int_{\xi}^{\omega} G(\tau_i, s)a_i(s)\Delta s = \max_{t\in[\xi,\omega]_{\mathbb{T}}} \int_{\xi}^{\omega} G(t, s)a_i(s)\Delta s.$$

Finally, we define

$$l = \min_{s\in[a,\sigma(b)]_{\mathbb{T}}} \frac{G(\sigma(\omega), s)}{G(\sigma(s), s)},$$

and let

$$\gamma = \min\{k, l\}. \tag{3.1}$$

For our construction, let $\mathcal{B} = \{x : [a, \sigma^2(b)]_{\mathbb{T}} \to \mathbb{R}\}$ with supremum norm $\|x\| = \sup\{|x(t)| : t \in [a, \sigma^2(b)]_{\mathbb{T}}\}$, and define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \Big\{x \in \mathcal{B} : x(t) \geqslant 0 \text{ on } [a, \sigma^2(b)]_{\mathbb{T}}, \text{ and } \min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} x(t) \geqslant \gamma\|x\|\Big\}.$$

We next define an integral operator $T : \mathcal{P} \to \mathcal{B}$, for $u \in \mathcal{P}$, by

(3.2)

$$\begin{aligned} Tu(t) = \lambda_1 \int_a^{\sigma(b)} & G(t, s_1)a_1(s_1)f_1\Big(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)a_2(s_2)\times \\ & \times f_2\Big(\lambda_3 \int_a^{\sigma(b)} G(\sigma(s_2), s_3)a_3(s_3)...\times \\ & \times f_{n-1}\Big(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)...\Delta s_3\Big)\Delta s_2\Big)\Delta s_1. \end{aligned}$$

Notice from $(A1), (A2)$, and (2.4) that, for $u \in \mathcal{P}$, $Tu(t) \geqslant 0$ on $[a, \sigma^2(b)]_{\mathbb{T}}$. Also, for $u \in \mathcal{P}$, we have from (2.5) that

$$\begin{aligned} Tu(t) \leqslant \lambda_1 \int_a^{\sigma(b)} & G(\sigma(s_1), s_1)a_1(s_1)f_1\Big(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)a_2(s_2)\times \\ & \times f_2\Big(\lambda_3 \int_a^{\sigma(b)} G(\sigma(s_2), s_3)a_3(s_3)...\times \\ & \times f_{n-1}\Big(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)...\Delta s_3\Big)\Delta s_2\Big)\Delta s_1. \end{aligned}$$

so that

(3.3)

$$\begin{aligned} \|Tu\| \leqslant \lambda_1 \int_a^{\sigma(b)} & G(\sigma(s_1), s_1)a_1(s_1)f_1\Big(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2)a_2(s_2)\times \\ & \times f_2\Big(\lambda_3 \int_a^{\sigma(b)} G(\sigma(s_2), s_3)a_3(s_3)...\times \\ & \times f_{n-1}\Big(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)...\Delta s_3\Big)\Delta s_2\Big)\Delta s_1. \end{aligned}$$

Next, if $u \in \mathcal{P}$, we have from (2.6), (3.1), and (3.3) that

$$\min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} Tu(t)$$

$$= \min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} \lambda_1 \int_a^{\sigma(b)} G(t,s_1)a_1(s_1)f_1\Big(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1),s_2)a_2(s_2)\times$$

$$\times f_2\Big(...f_{n-1}\Big(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)...\Delta s_3\Big)\Delta s_2\Big)\Delta s_1$$

$$\geqslant \lambda_1\gamma \int_a^{\sigma(b)} G(\sigma(s_1),s_1)a_1(s_1)f_1\Big(\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1),s_2)a_2(s_2)\times$$

$$\times f_2\Big(...f_{n-1}\Big(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}),s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)...\Delta s_3\Big)\Delta s_2\Big)\Delta s_1$$

$$\geqslant \gamma\|Tu\|.$$

Consequently, $T : \mathcal{P} \to \mathcal{P}$. In addition, the standard arguments shows that T is completely continuous.

By the remarks in Section 2, we seek suitable fixed points of T belonging to the cone $\mathcal{P}$.

For our first result, define positive numbers L_1 and L_2, by

$$L_1 := \max_{1\leqslant i\leqslant n}\left\{\Big[\gamma \int_\xi^\omega G(\tau_i,s)a_i(s)\Delta s f_{i\infty}\Big]^{-1}\right\},$$

and

$$L_2 := \min_{1\leqslant i\leqslant n}\left\{\Big[\int_a^{\sigma(b)} G(\sigma(s),s)a_i(s)\Delta s f_{i0}\Big]^{-1}\right\}.$$

Theorem 3.1. *Assume that conditions* (A1)-(A3) *are satisfied. Then, for each* $\lambda_1,\lambda_2,...,\lambda_n$ *satisfying*

$$L_1 < \lambda_i < L_2,\ 1 \leqslant i \leqslant n \tag{3.4}$$

there exists an n-tuple $(u_1,u_2,...,u_n)$ satisfying (1.1), (1.2) *such that $u_i(t) > 0$ on $(a,\sigma^2(b))_{\mathbb{T}}$, $1 \leqslant i \leqslant n$.*

Proof. Let λ_j, $1 \leqslant j \leqslant n$, be as in (3.4). And let $\epsilon > 0$ be chosen such that

$$\max_{1\leqslant i\leqslant n}\left\{\Big[\gamma \int_\xi^\omega G(\tau_i,s)a_i(s)\Delta s(f_{i\infty}-\epsilon)\Big]^{-1}\right\} \leqslant \min_{1\leqslant j\leqslant n}\lambda_j,$$

and

$$\max_{1\leqslant j\leqslant n}\lambda_j \leqslant \min_{1\leqslant i\leqslant n}\left\{\Big[\int_a^{\sigma(b)} G(\sigma(s),s)a_i(s)\Delta s(f_{i0}+\epsilon)\Big]^{-1}\right\}.$$

We seek fixed points of the completely continuous operator $T : \mathcal{P} \to \mathcal{P}$ defined by (3.2).

Now, from the definitions of f_{i0}, $1 \leqslant i \leqslant n$, there exists an $H_1 > 0$ such that, for each $1 \leqslant i \leqslant n$,

$$f_i(x) \leqslant (f_{i0}+\epsilon)x,\ 0 < x \leqslant H_1.$$

Let $u \in \mathcal{P}$ with $\|u\| = H_1$. We first have from (2.5) and choice of ϵ, for $a \leqslant s_{n-1} \leqslant \sigma(b)$,

$$\begin{aligned}
&\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\
&\leqslant \lambda_n \int_a^{\sigma(b)} G(\sigma(s_n), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\
&\leqslant \lambda \int_a^{\sigma(b)} G(\sigma(s_n), s_n) a_n(s_n) (f_{n0} + \epsilon) u(\sigma(s_n)))\Delta s_n \\
&\leqslant \lambda_n \int_a^{\sigma(b)} G(\sigma(s_n), s_n) a_n(s_n) \Delta s_n (f_{n0} + \epsilon)\|u\| \\
&\leqslant \|u\| \\
&= H_1.
\end{aligned}$$

It follows in a similar manner from (2.5) and choice of ϵ that, for $a \leqslant s_{n-2} \leqslant \sigma(b)$,

$$\begin{aligned}
&\lambda_{n-1} \int_a^{\sigma(b)} G(\sigma(s_{n-2}), s_{n-1}) a_{n-1}(s_{n-1}) \times \\
&\times f_{n-1}\Big(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n\Big)\Delta s_{n-1} \\
&\leqslant \lambda_{n-1} \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_{n-1}) a_{n-1}(s_{n-1}) \Delta s_{n-1} (f_{n-1,0} + \epsilon)\|u\| \\
&\leqslant \|u\| \\
&= H_1.
\end{aligned}$$

Continuing with this bootstrapping, we reach, for $a \leqslant t \leqslant \sigma^2(b)$,

$$\lambda_1 \int_a^{\sigma(b)} G(t, s_1) a_1(s_1) f_1(...f_n(u(\sigma(s_n)))\Delta s_n ...)\Delta s_1 \leqslant H_1,$$

so that, for $a \leqslant t \leqslant \sigma^2(b)$,

$$Tu(t) \leqslant H_1,$$

or

$$\|Tu\| \leqslant H_1 = \|u\|.$$

If we set

$$\Omega_1 = \{x \in \mathcal{B} : \|x\| < H_1\},$$

then

$$\|Tu\| \leqslant \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.5}$$

Next, from the definitions of $f_{i\infty}$, $1 \leqslant i \leqslant n$, there exists $\overline{H}_2 > 0$ such that, for each $1 \leqslant i \leqslant n$,

$$f_i(x) \geqslant (f_{i\infty} - \epsilon)x, \ x \geqslant \overline{H}_2.$$

Let

$$H_2 = \max\Big\{2H_1, \frac{\overline{H}_2}{\gamma}\Big\}.$$

Let $u \in \mathcal{P}$ and $\|u\| = H_2$. Then,

$$\min_{t\in[\xi,\sigma(\omega)]_{\mathbb{T}}} u(t) \geqslant \gamma\|u\| \geqslant \overline{H}_2.$$

Consequently, from (2.6) and choice of ϵ, for $a \leqslant s_{n-1} \leqslant \sigma(b)$, we have that

$$\begin{aligned}
&\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n \\
&\geqslant \lambda_n \int_\xi^{\omega} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n \\
&\geqslant \lambda_n \int_\xi^{\omega} G(\tau_n, s_n)a_n(s_n)(f_{n\infty} - \epsilon)(u(\sigma(s_n)))\Delta s_n \\
&\geqslant \gamma\lambda_n \int_\xi^{\omega} G(\tau_n, s_n)a_n(s_n)\Delta s_n(f_{n\infty} - \epsilon)\|u\| \\
&\geqslant \|u\| \\
&= H_2.
\end{aligned}$$

It follows similarly from (2.6) and choice of ϵ, for $a \leqslant s_{n-2} \leqslant \sigma(b)$,

$$\begin{aligned}
&\lambda_{n-1} \int_a^{\sigma(b)} G(\sigma(s_{n-2}), s_{n-1})a_{n-1}(s_{n-1})\times \\
&\times f_{n-1}\Big(\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n)a_n(s_n)f_n(u(\sigma(s_n)))\Delta s_n\Big)\Delta s_{n-1} \\
&\geqslant \gamma\lambda_{n-1} \int_\xi^{\omega} G(\tau_{n-1}, s_{n-1})a_{n-1}(s_{n-1})\Delta s_{n-1}(f_{n-1,\infty} - \epsilon)\|u\| \\
&\geqslant \|u\| \\
&= H_2.
\end{aligned}$$

Again, using a bootstrapping, we reach

$$Tu(\tau_1) = \lambda_1 \int_a^{\sigma(b)} G(\tau_1, s_1)a_1(s_1)f_1(...f_n(u(\sigma(s_n)))\Delta s_n...)\Delta s_1 \geqslant \|u\| = H_2,$$

so that, $\|Tu\| \geqslant \|u\|$. So if we set

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_2\},$$

then

$$\|Tu\| \geqslant \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.6}$$

Applying Theorem 2.1 to (3.5) and (3.6), we obtain that T has a fixed point $u \in \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$. As such, setting $u_1 = u_{n+1} = u$, we obtain a positive solution

$(u_1, u_2, ..., u_n)$ of (1.1)-(1.2) given iteratively by

$$u_j(t) = \lambda_j \int_a^{\sigma(b)} G(t,s)a_j(s)f_j(u_{j+1}(\sigma(s)))\Delta s, \;\; j = n, n-1, ..., 1.$$

The proof is complete. □

Prior to our next result, let ξ_i, $1 \leqslant i \leqslant n$, be defined by

$$\int_a^{\sigma(b)} G(\xi_i, s)a_i(s)\Delta s = \max_{t\in[a,\sigma^2(b)]_{\mathbb{T}}} \int_a^{\sigma(b)} G(t,s)a_i(s)\Delta s.$$

Then, we define positive numbers L_3 and L_4 by

$$L_3 := \max_{1\leqslant i\leqslant n} \left\{ \left[\gamma \int_\xi^\omega G(\tau_i, s)a_i(s)\Delta s f_{i0} \right]^{-1} \right\},$$

and

$$L_4 := \max_{1\leqslant i\leqslant n} \left\{ \left[\int_a^{\sigma(b)} G(\xi_i, s)a_i(s)\Delta s f_{i\infty} \right]^{-1} \right\}.$$

THEOREM 3.2. *Assume that conditions* $(A1)$*-*$(A3)$ *are satisfied. Then, for each* $\lambda_1, \lambda_2, ..., \lambda_n$ *satisfying*

(3.7) $$L_3 < \lambda_i < L_4, \;\; 1 \leqslant i \leqslant n$$

there exists an n-tuple $(u_1, u_2, ..., u_n)$ *satisfying* (1.1)-(1.2) *such that* $u_i(t) > 0$ *on* $(a, \sigma^2(b))_{\mathbb{T}}$*,* $1 \leqslant i \leqslant n$*.*

PROOF. Let λ_j, $1 \leqslant j \leqslant n$ be as in (3.7). And let $\epsilon > 0$ be chosen such that

$$\max_{1\leqslant i\leqslant n} \left\{ \left[\gamma \int_\xi^\omega G(\tau_i, s)a_i(s)\Delta s (f_{i0} - \epsilon) \right]^{-1} \right\} \leqslant \min_{1\leqslant j\leqslant n} \lambda_j,$$

and

$$\max_{1\leqslant j\leqslant n} \lambda_j \leqslant \min_{1\leqslant i\leqslant n} \left\{ \left[\int_a^{\sigma(b)} G(\xi_i, s)a_i(s)\Delta s (f_{i\infty} + \epsilon) \right]^{-1} \right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (3.2).

From the definition of f_{i0}, $1 \leqslant i \leqslant n$ there exists $\overline{H}_3 > 0$ such that, for each $1 \leqslant i \leqslant n$,

$$f_i(x) \geqslant (f_{i0} - \epsilon)x, \;\; 0 < x \leqslant \overline{H}_3.$$

Also, from the definitions of f_{i0}, it follows that $f_{i0}(0) = 0$, $1 \leqslant i \leqslant n$, and so there exist $0 < K_n < K_{n-1} < ... < K_2 < \overline{H}_3$ such that

$$\lambda_i f_i(t) \leqslant \frac{K_{i-1}}{\int_a^{\sigma(b)} G(\xi_i, s)a_i(s)\Delta s}, \;\; t \in [0, K_i]_{\mathbb{T}}, \;\; 3 \leqslant i \leqslant n,$$

and

$$\lambda_2 f_2(t) \leqslant \frac{\overline{H}_3}{\int_a^{\sigma(b)} G(\xi_2, s)a_2(s)\Delta s}, \;\; t \in [0, K_2]_{\mathbb{T}}.$$

Choose $u \in \mathcal{P}$ with $\|u\| = K_n$. Then, we have

$$\begin{aligned}
&\lambda_n \int_a^{\sigma(b)} G(\sigma(s_{n-1}), s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\
&\leqslant \lambda_n \int_a^{\sigma(b)} G(\xi_n, s_n) a_n(s_n) f_n(u(\sigma(s_n)))\Delta s_n \\
&\leqslant \frac{\int_a^{\sigma(b)} G(\xi_n, s_n) a_n(s_n) K_{n-1}\Delta s_n}{\int_a^{\sigma(b)} G(\xi_n, s_n) a_n(s_n)\Delta s_n} \\
&\leqslant K_{n-1}.
\end{aligned}$$

Bootstrapping yields the standard iterative pattern, and it follows that

$$\lambda_2 \int_a^{\sigma(b)} G(\sigma(s_1), s_2) a_2(s_2) f_2(\ \ldots\ )\Delta s_2 \leqslant \overline{H}_3.$$

Then,

$$\begin{aligned}
Tu(\tau_1) &= \lambda_1 \int_a^{\sigma(b)} G(\tau_1, s_1) a_1(s_1) f_1(\lambda_2\ \ldots\ )\Delta s_1 \\
&\geqslant \lambda_1 \gamma \int_\xi^\omega G(\tau_1, s_1) a_1(s_1)(f_{1,0} - \epsilon)\|u\|\Delta s_1 \\
&\geqslant \|u\|.
\end{aligned}$$

So, $\|Tu\| \geqslant \|u\|$. If we put

$$\Omega_1 = \{x \in \mathcal{B} : \|x\| < K_n\},$$

then

$$\|Tu\| \geqslant \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1. \tag{3.8}$$

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that f_i, $1 \leqslant i \leqslant n$, is unbounded at ∞.

For each $1 \leqslant i \leqslant n$, set

$$f_i^*(x) = \sup_{a \leqslant s \leqslant x} f_i(s).$$

Then, it is straightforward that, for each $1 \leqslant i \leqslant n$, f_i^* is a nondecreasing real-valued function, $f_i \leqslant f_i^*$, and

$$\lim_{x\to\infty} \frac{f_i^*(x)}{x} = f_{i\infty}.$$

Next, by definition of $f_{i\infty}$, $1 \leqslant i \leqslant n$, there exists $\overline{H}_4$ such that, for each $1 \leqslant i \leqslant n$,

$$f_i^*(x) \geqslant (f_{i\infty} + \epsilon)x, \ x \geqslant \overline{H}_4.$$

It follows that there exists $H_4 > \max\{2\overline{H}_3, \overline{H}_4\}$ such that, for each $1 \leqslant i \leqslant n$,

$$f_i^*(x) \leqslant f_i^*(H_4), \ 0 < x \leqslant H_4.$$

Choose $u \in \mathcal{P}$ with $\|u\| = H_4$. Then, using the usual bootstrapping argument, we have

$$\begin{aligned} Tu(t) &= \lambda_1 \int_a^{\sigma(b)} G(t,s_1)a_1(s_1)f_1(\lambda_2 \ldots)\Delta s_1 \\ &\leqslant \lambda_1 \int_a^{\sigma(b)} G(t,s_1)a_1(s_1)f_1^*(\lambda_2 \ldots)\Delta s_1 \\ &\leqslant \lambda_1 \int_a^{\sigma(b)} G(\xi_1,s_1)a_1(s_1)f_1^*(H_4)\Delta s_1 \\ &\leqslant \lambda_1 \int_a^{\sigma(b)} G(\xi_1,s_1)a_1(s_1)\Delta s_1(f_{1\infty}+\epsilon)H_4 \\ &\leqslant H_4 \\ &= \|u\|, \end{aligned}$$

and so $\|Tu\| \leqslant \|u\|$. So, if we let

$$\Omega_2 = \{x \in \mathcal{B} : \|x\| < H_4\},$$

then

$$\|Tu\| \leqslant \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2. \tag{3.9}$$

Application of part (ii) of Theorem 2.1 yields a fixed point u of T belonging to $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which in turn, with $u_1 = u_{n+1} = u$, yields an n-tuple $(u_1, u_2, ..., u_n)$ satisfying (1.1)-(1.2) for the chosen values of λ_i, $1 \leqslant i \leqslant n$. The proof is complete. □

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