BULLETIN OF INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN 1840-4367
Vol. 2(2012), 43-46

Former
BULLETIN OF SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

TWO EXTENSIONS OF STEINHAUS'S THEOREM

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Communicated by

Abstract. In 1920 H. Steinhaus [Sur les distances des points de mesure positive, Fundamenta Mathematicae 1 (1920) 93-104.] proved the following result: "Let A be a Lebesgue measurable set of positive measure. Then there exist at least two points in A such that the distance between them is a rational number".

In this paper we shall prove that there exists a sequence $(x_n)_{n\geqslant 1}$ of different points in A such that the distance between any two of them is a rational number. Further, we shall extend our result to the case when A is a set with the Baire property (non-necessarily Lebesgue measurable).

1. Introduction

The set of rational numbers will be denoted by $\mathbb{Q}$ and the set of real numbers by $\mathbb{R}$.

Let λ be Lebesgue measure on the set of real numbers $\mathbb{R}$. If $(A_n)_{n\geqslant 1}$ is a sequence of Lebesgue measurable sets in $\mathbb{R}$, then we have the following inequality:

$$\lambda(\underline{\lim}_{n\to\infty} A_n) \leqslant \underline{\lim}_{n\to\infty} \lambda(A_n).$$

For the inequality

$$\overline{\lim}_{n\to\infty} \lambda(A_n) \leqslant \lambda(\overline{\lim}_{n\to\infty} A_n)$$

we must suppose that $\lambda\big(\bigcup_{i=n}^{\infty} A_n\big) < \infty$ for at least one value of n (see [**6**, p. 40]).

Example 1.1. For a family of intervals $I_n = [\,n, n+1)$, $n = 0, 1, \ldots$, we have: $\overline{\lim}_{n\to\infty} \lambda(A_n) = 1$ and $\lambda(\overline{\lim}_{n\to\infty} A_n) = 0$.

2000 *Mathematics Subject Classification.* Primary 28A05; Secondary 54E52.

Key words and phrases. inequality, Lebesgue measure, Baire category.

The first author was partially supported by MNZZS Grant, No. 174002, Serbia. The second author was partially supported by MNZZS Grant, No. 174017, Serbia.

In [**1**] the first author presented the following general inequality for Lebesgue measure and gave some of its applications.

THEOREM 1.1. (I. Aranđelović [**1**]) *Let A be a measurable set of positive measure and $(x_n)_{n\geqslant 1}$ be a bounded sequence of real numbers. Then*

$$\lambda(A) \leqslant \lambda(\overline{\lim_{n\to\infty}}(x_n + A)).$$

Further applications of this inequality can be found in [**3**], [**4**] and [**2**].

$A \subseteq \mathbf{R}$ is of first Baire category if it is a countable union of nowhere dense sets. Otherwise, A is of second Baire category. If A is of second Baire category, then there exists an open set $\mathcal{O}$ and a first Baire category set P such that

$$A = P \Delta \mathcal{O},$$

where Δ denotes the symmetric difference.

In 1920 H. Steinhaus [**8**] proved the following result:

THEOREM 1.2 (H. Steinhaus [**8**]). *Let A be a Lebesgue measurable set of positive measure. Then there exist at least two points in A such that the distance between them is a rational number.*

In this paper we shall prove that there exist sequences $(x_n)_{n\geqslant 1}$ of different points in A such that the distance between any two of them is a rational number. Further we extend our result to the case when A is a set with the Baire property (non-necessarily Lebesgue measurable).

2. Main Results

Now we present our main result.

THEOREM 2.1. *Let A be a Lebesgue measurable set of positive measure. Then there exists a sequence $(x_n)_{n\geqslant 1}$ of different points in A such that the distance between any two of them is a rational number.*

PROOF. Let $(q_n)_{n\geqslant 1}$ be an arbitrary bounded sequence of rational numbers whose terms are pairwise different. From

$$0 < \lambda(A) \leqslant \lambda(\overline{\lim_{n\to\infty}}(A + q_n))$$

it follows that there exists $x_* \in \overline{\lim_{n\to\infty}}(A + q_n)$. Thus there exists an increasing sequence of positive integers $(n_j)_{j\geqslant 1}$ and a sequence of points $(p_n)_{n\geqslant 1} \subseteq A$ such that

$$p_j + q_{n_j} = x_*,$$

for any positive integer j.

Hence, for each $i \neq j$ we obtain $|p_i - p_j| = |q_{n_i} - q_{n_j}| \neq 0$, so $p_i \neq p_j$ for $i \neq j$, and

$$|p_i - p_j| = |q_{n_i} - q_{n_j}| \in \mathbb{Q}.$$

□

Now we need the following lemma.

LEMMA 2.1. *Let A be a set which has the Baire property and $(x_n)_{n\geqslant 1}$ be a bounded sequence of real numbers. Then the set $\overline{\lim\limits_{n\to\infty}}\,(x_n + A)$ is nonempty.*

PROOF. In [**5**] it was proved that there exists an open interval I which contains zero such that for any $(y_n)_{n\geqslant 1} \subseteq I$ there exists $a \in A$ such that for all n we have

$$a + y_n \in A.$$

Also, the sequence $(-x_n)_{n\geqslant 1}$ has a cluster point $x_* \in \mathbb{R}$, because it is bounded. So, there exists the subsequence $(x_{n_j})_{j\geqslant 1}$ such that

$$(-x_{n_j} - x_*)_{j\geqslant 1} \subseteq I,$$

which implies that there exists $a_* \in A$ such that

$$(-x_{n_j} - x_* + a_*)_{j\geqslant 1} \subseteq A.$$

It follows that

$$(a_* - x_*) \in x_{n_j} + A,$$

for any positive integer j. Hence

$$\overline{\lim_{n\to\infty}}\,(x_n + A) \neq \emptyset.$$

□

The proof of next result is essentially the same as the proof of Theorem 2.1. For the convenience of the reader, we present it here.

THEOREM 2.2. *Let A be a set which has the Baire property. Then there exists a sequence $(x_n)_{n\geqslant 1}$ of different points in A such that the distance between any two of them is a rational number.*

PROOF. Let $(q_n)_{n\geqslant 1}$ be an arbitrary bounded sequence of rational numbers whose terms are different. Then the set $\overline{\lim\limits_{n\to\infty}}\,(A + q_n)$ is nonempty, because set A has the Baire property.

So, there exist an increasing sequence of positive integers $(n_j)_{j\geqslant 1}$ and a sequence of points $(p_n)_{n\geqslant 1} \subseteq A$ such that

$$p_j + q_{n_j} = x_*,$$

for any positive integer j.

Hence, for each $i \neq j$ we obtain $|p_i - p_j| = |q_{n_i} - q_{n_j}| \neq 0$, so $p_i \neq p_j$ for $i \neq j$, and

$$|p_i - p_j| = |q_{n_i} - q_{n_j}| \in \mathbb{Q}.$$

□

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(Received by editors 04.12.2011, in revision form 06.02.2012. Available online 13.02.2012)

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