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THE AVERAGE LOWER INDEPENDENCE NUMBER OF TOTAL GRAPHS

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Abstract

In communication networks, "vulnerability" indicates the resistance of a network to disruptions in communication after a breakdown of some processors or communication links. We may use graphs to model networks, as graph theoretical parameters can be used to describe the stability and reliability of communication networks If we think of a graph as modeling a network, the average lower independence number of a graph is one measure of graph vulnerability. For a vertex v of a graph $G = (V, E)$, the lower independence number $i_v(G)$ of G relative to v is the minimum cardinality of a maximal independent set of G that contains v. The average lower independence number of G, denoted by $i_{av}(G)$, is the value $i_{av}(G) = \frac{1}{|V(G)|} \sum_{v \epsilon V(G)} i_v(G)$. In this paper, we defined and examined this parameter and considered the average lower independence number of special graphs and theirs total graphs.

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1 Introduction

In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. When a network begins losing stations or communication links there is, eventually, a loss in its effectiveness. Thus, a communication network must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. If we think of a graph as modeling a network, then there are many

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graph theoretical parameters such as connectivity, toughness, integrity, domination and its variations. Domination and its variations in graphs are now well studied. In this paper, we introduce and study the concept of average lower independence number in graphs, a concept closely related to the problem of finding large independent sets in graphs.

A graph G is denoted by $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are vertex and edge sets of G, respectively. n denotes the number of vertices and m denotes the number of edges of the graph G, and let v be a vertex in V. The open neighborhood of v is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, its open neighborhood $N(S) = \cup_{v\in S} N(v)$ and its closed neighborhood $N[S] = N(S) \cup S$.

A set S is dominating set of G if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to at least one vertex of S. The *dominating number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. An independent set of vertices of a graph G is a set of vertices of G whose elements are pair wise nonadjacent. The *independence number* $\beta(G)$ of G is the maximum cardinality among all independent sets of vertices of G, while the *independent domination number* (also called the *lower independence number*) $i(G)$ of G is the minimum cardinality of a maximal independent set of G [4,7].

For a vertex v of a graph $G = (V, E)$, the lower independence number $i_v(G)$ of G relative to v is the minimum cardinality of a maximal independent set of G that contains v. The average lower independence number of G, denoted by $i_{av}(G)$, is the value $i_{av}(G) = \frac{1}{|V(G)|}\sum_{v \epsilon V(G)} i_v(G)$. Throughout to this paper $i_v(G)$-set is refer to maximal independence set including vertex v [4,7].

The next section contains results on the average lower independence number of graph G, In Section 3, we formulates average lower independence number of total graph of some basic graphs.

2 Basic Results

In this section, we will review some of the known result on average lower independence number.

Theorem 2.1 [4, 7] *For every vertex v in a graph,*

$$a)\ i(G) \leq i_v(G) \leq \beta(G)$$
$$b)\ i(G) \leq i_{av}(G) \leq \beta(G)$$

Theorem 2.2 [4] *For any graph G of order n. Then,*

$$i_{av}(G) \leq \beta(G) - \frac{i(G)(\beta(G) - i(G))}{n}.$$

Theorem 2.3 [7] *If T is a tree of order $n \geq 2$. Then,*

$$i_{av}(G) \leq n - 2 + \frac{2}{n}.$$

Theorem 2.4 [2] *Let G_1 and G_2 be two connected graphs and $\beta(G_1) < \beta(G_2)$. Then,*

$$i_{av}(G_1) + i_{av}(G_2) < 2\beta(G_2)$$

Theorem 2.5 [2] *Let G_1 and G_2 be two connected graphs and $i(G_1) < i(G_2)$. Then,*

$$2i(G_1) < i_{av}(G_1) + i_{av}(G_2)$$

Theorem 2.6 [1] *For two graphs G_1 and G_1 of order m and n, respectively,*

$$i_{av}(G_1 + G_2) = \frac{i_{av}(G_1).m + i_{av}(G_2).n}{m+n}.$$

Theorem 2.7 [1] *For two graphs G_1 and G_1 of order m and n, respectively,*

$$i_{av}(G_1 + G_2) \leq \frac{\beta(G_1).m + \beta(G_2).n}{m+n}.$$

Theorem 2.8 [1] *For complete graph K_n of order n and for any graph G of order m,*

$$i_{av}(GoK_n) = m.$$

3 Average Lower Independence Number Of Total Graphs

In this section, we give some results on the average lower independence number of $T(G)$ total graph are calculated.

Definition 3.1 [9] *The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. The total graph $T(G)$ of the graph $G = (V(G), E(G))$, has vertex set $V(G) \cup E(G)$, and two vertices of $T(G)$ are adjacent whenever they are neighbors in G. It is easy to see that $T(G)$ always contains both G and Line graph $L(G)$ as a induced subgraphs. The total graph is the largest graph that is formed by the adjacent relations of elements of a graph. It is important from this respect.*

Theorem 3.1 *The average lower independence number of $T(K_{1,n})$ with order of $(2n+1)$ is defined as $i_{av}(K_{1,n}) = \frac{2n^2+1}{2n+1}$.*

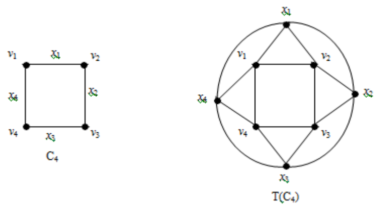

Figure 1

Proof. The number of vertices of graphs $K_{1,n}$ and $T(K_{1,n})$ are $|V(K_{1,n})| = n+1$ and $|V(T(K_{1,n}))| = 2n + 1$, respectively. Let $T(K_{1,n})$ be G. If we think that the vertex-set of graph G be $V(G) = V_1(G) \cup V_2(G) \cup V_3(G)$ where,

$V_1(G)$: The set contains center vertex with degree of $2n$ of the graph G.
$V_2(G)$: The set contains n vertices except center vertex of the star graph $K_{1,n}$ in graph G.
$V_3(G)$: The set contains n new vertices with degree of $(n+1)$ which are obtained by definition of total graph.

Then, it is easily calculated that the average lower independence number of graph G. When we calculate the for all v vertices in a graph G, we should examine the vertices in three cases.

Case 1. Let v be the vertex of the $V_1(G)$. The vertex v is the center vertex with n degree of the star graph $K_{1,n}$. Center vertex v adjacent to other $2n$ vertices due to structural of graph G. Consequently, being v the center vertex,$i_v(G) = 1$.

Case 2. Let v be the vertex of the $V_2(G)$. As forming the maximal independence set including v, we can use 2 distinct ways:

i) We must not take center vertex and a vertex in $V_3(G)$ that adjacent vertex v. Then, we take n vertices at the extremity of the leaves of the graph $K_{1,n}$. There are no edges between these taken n vertices, thus set is an independence set, however maximal independence set. We have to repeat this process for n vertices with degree 2. Hence, $i_v(G) = n$.

ii) In $i_v(G)$-independence sets, we can take the only one vertex in $V_3(G)$ which doesn't adjacent to the vertex v. The vertices of $V_3(G)$ are adjacent to each other. And also the vertex u is adjacent to a vertex v in $V_2(G)$. Now there are two vertices in $i_v(G)$-set. To form the whole $i_v(G)$-sets, including vertices except u and v, we should add the remaining $n-2$ vertices of $V_2(G)$ in the lower independence set. Now the $i_v(G)$-set there were two vertices. When we add the remaining $n-2$ vertices to the $i_v(G)$-set, consequently, there are totally n vertices in the $i_v(G)$-set . We have to repeat this process for the whole n vertices with degree 2. Then, we get $i_v(G) = n$.

From ***i*** and ***ii***; whenever $u \in V_2(G)$, we have $i_v(G) = n$.

Case 3. Let v be the vertex of $V_3(G)$. The $i_v(G)$-set, which includes vertex v doesn't include the other vertices of $V_3(G)$, because vertex v is adjacent to the other vertices of $V_3(G)$. Thus, $i_v(G)$ can't include the center vertex, which is adjacent to the vertex v, and one of the vertices with degree 2 in $V_2(G)$. Consequently; to have the $i_v(G)$-independence set; we should add the remaining $n-1$ vertices in $V_2(G)$ to the $i_v(G)$-set. In the $i_v(G)$-set, with vertex v; we have taken $n-1$ vertices. Hence, the $i_v(G)$-independence set has totally n vertices. We have to repeat this process for n vertices of $V_3(G)$. Thus, we get $i_v(G) = n$. Consequently, by *case1,case2* and *case3*, we have;

$$
\begin{aligned}
i_{av}(G) &= \tfrac{1}{|V(G)|}\left(\textstyle\sum_{v\epsilon V_1(G)} i_v(G) + \sum_{v\epsilon V_2(G)} i_v(G) + \sum_{v\epsilon V_3(G)} i_v(G)\right) \\
&= \tfrac{1}{2n+1}\left(1 + \sum_{v\epsilon V_2(G)}^{n} n + \sum_{v\epsilon V_3(G)}^{n} n\right) \\
&= \tfrac{1}{2n+1}(1 + n^2 + n^2) \\
&= \tfrac{2n^2+1}{2n+1}
\end{aligned}
$$

The proof is completed. □

Theorem 3.2 *Let $T(C_n)$ be the total graph of C_n. Then,*

$$
i_{av}(T(C_n)) = \begin{cases} \frac{2n}{5} & , \text{ if } n \bmod 5 \equiv 0 \\ \lfloor \frac{2n}{5} \rfloor + 1 & , \text{ otherwise.} \end{cases}
$$

Proof. The number of vertices of graphs C_n and $T(C_n)$ are $V|C_n| = n$ and $V|T(C_n)| = 2n$, respectively. Graph $T(C_n)$ is a 4-regular graph . Let $T(C_n)$ be G. When we take any vertex of graph G, it is adjacent 4 vertices in graph G. Hence, 5 vertices are dominated by the vertex v, vertex v and the other 4 vertices adjacent to v. We have to repeat this process every 5 vertices. Therefore, we have 5 cases according to the number of vertices of G.

Case1: ((2n) mod 5 ≡ 0): If $n(mod 5) \equiv 0$, $i_v(G)$-set has $\frac{2n}{5}$ vertices. So, we get $i_v(G) = \frac{2n}{5}$. Thus, $i_{av}(G) = \frac{1}{|V(G)|} \sum_{v\epsilon V(G)}^{2n} \frac{2n}{5} = \frac{2n}{5}$. (1)

Case2: ((2n) mod 5 ≡ 1): When number of vertices graph G is $5k+1$, then $\lfloor \frac{2n}{5} \rfloor$ subgraph has 5 vertices of graph G. We have to add center vertices of the induced subgraph of $\lfloor \frac{2n}{5} \rfloor$ in graph G to $i_v(G)$-sets, then finally the only one vertex remaining in graph G. Thus, ve have $i_v(G) = \lfloor \frac{2n}{5} \rfloor + 1$. (2)

Case3: ((2n) mod 5 ≡ 2): When number of vertices graph G is $5k+2$, then $\lfloor\frac{2n}{5}\rfloor$ subgraph has 5 vertices of graph G. We have to add center vertices of the induced subgraph of $\lfloor\frac{2n}{5}\rfloor$ in graph G to $i_v(G)$-sets, then as being the remaining 2 vertices adjacent to each other, finally the only one of the 2 vertices remaining in graph G. Thus, the set we have is an set $i_v(G)$. Then, we have $i_v(G) = \lfloor\frac{2n}{5}\rfloor + 1$. (3)

Case4: ((2n) mod 5 ≡ 3): When number of vertices graph G is $5k+3$, then $\lfloor\frac{2n}{5}\rfloor$ subgraph has 5 vertices of graph G. We have to add center vertices of the induced subgraph of $\lfloor\frac{2n}{5}\rfloor$ in graph G to $i_v(G)$-sets, then as being the remaining 3 vertices adjacent to each other, finally the only one of the 3 vertices remaining in graph G. Thus, the set we have is an $i_v(G)$-set.Then, we have $i_v(G) = \lfloor\frac{2n}{5}\rfloor + 1$. (4)

Case5: ((2n) mod 5 ≡ 4): When number of vertices graph G is $5k+4$, then $\lfloor\frac{2n}{5}\rfloor$ subgraph has 5 vertices of graph G. We have to add center vertices of the induced subgraph of $\lfloor\frac{2n}{5}\rfloor$ in graph G. Finally remaining graph structural which has 4 vertices is maximum vertex degree with 3. We show it to Figure2.

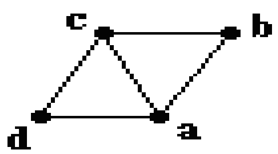

Figure 2

We take one vertices which vertex degree with 3 from structural Figure2. Thus, the set we have is an $i_v(G)$-set. Then, we have $i_v(G) = \lfloor\frac{2n}{5}\rfloor + 1$. (5)

From (2),(3),(4) and (5), we have;

$$i_{av}(G) = \frac{1}{|V(G)|}\sum_{v \epsilon V(G)}^{2n}(\lfloor\frac{2n}{5}\rfloor + 1)$$

$$i_{av}(G) = \frac{1}{2n}.2n.(\lfloor\frac{2n}{5}\rfloor + 1) \Rightarrow i_{av}(G) = \lfloor\frac{2n}{5}\rfloor + 1 \qquad (6)$$

From (1) and (6), the proof is completed. □

Theorem 3.3 *Let $T(P_n)$ be the total graph of P_n. Then,*

$$i_{av}(T(P_n)) = \begin{cases} \frac{1}{2n-1}(\lceil\frac{2n-1}{5}\rceil.(2n-2)+(2n-1)) & , \textit{if } (2n-1)\, mod\, 5 \equiv 0 \\ \lceil\frac{2n-1}{5}\rceil & , \textit{if } (2n-1)\, mod\, 5 \equiv 1 \\ \frac{1}{2n-1}(\lfloor\frac{2n-1}{5}\rfloor+\lceil\frac{2n-1}{5}\rceil.(2n-1)) & , \textit{if } (2n-1)\, mod\, 5 \equiv 2 \\ \frac{1}{2n-1}(\lceil\frac{2n-1}{5}\rceil.(2n-4)+(2n-1)) & , \textit{if } (2n-1)\, mod\, 5 \equiv 3 \\ \frac{1}{2n-1}(\lceil\frac{2n-1}{5}\rceil.(2n-3)+(2n-1)) & , \textit{if } (2n-1)\, mod\, 5 \equiv 4 \end{cases}$$

Proof. The number of vertices of P_n and $T(P_n)$ are $|V(P_n)| = n$ and $|V(T(P_n))| = 2n-1$, respectively. Let $T(P_n)$ be G and let v be the vertex of $T(P_n)$. The independence sets of $T(P_n)$, $i_v(G)$, are also the independence domination sets of $T(P_n)$. For the value $i_{av}(G)$, the $i_v(G)$ independence sets for every $v \in V(T(P_n))$ should be found including v, and also at least 3, at most 5 vertices can be dominated. Therefore, we have 5 cases according to the number of vertices of $T(P_n)$.

Case1: ((2n-1) mod 5 $\equiv$ 0): The values n such as $(2n-1)mod\ 5 \equiv 0$; firstly, It's seen that the $i_v(G)-$sets have $\lceil\frac{2n-1}{5}\rceil$ vertices of graph G. Nevertheless, $i_v(G)-$sets have $\lceil\frac{2n-1}{5}\rceil+1$ vertices for remaining whole $(2n-1)-\lceil\frac{2n-1}{5}\rceil$ vertices. Therefore;

$$\begin{aligned} i_{av}(G) &= \frac{1}{2n-1}.[\lceil\frac{2n-1}{5}\rceil.\lceil\frac{2n-1}{5}\rceil+((2n-1)-\lceil\frac{2n-1}{5}\rceil).(\lceil\frac{2n-1}{5}\rceil+1)] \\ &= \frac{1}{2n-1}[\lceil\frac{2n-1}{5}\rceil^2+(2n-1).\lceil\frac{2n-1}{5}\rceil+(2n-1)-\lceil\frac{2n-1}{5}\rceil^2-\lceil\frac{2n-1}{5}\rceil] \\ &= \frac{1}{2n-1}(\lceil\frac{2n-1}{5}\rceil.(2n-2)+(2n-1)). \end{aligned}$$

Case2: ((2n-1) mod 5 $\equiv$ 1): The cardinality of $i_v(G)-$sets are always same for every vertices of any graph G, and equals to $\lceil\frac{2n-1}{5}\rceil$. Then, we have;

$$i_{av}(G) = \frac{1}{2n-1}.(2n-1).\lceil\frac{2n-1}{5}\rceil = \lceil\frac{2n-1}{5}\rceil.$$

Case3: ((2n-1) mod 5 $\equiv$ 2): $i_v(G)-$sets have $\lceil\frac{2n-1}{5}\rceil+1$ vertices for $\lfloor\frac{2n-1}{5}\rfloor$ vertices of graph G. Moreover, $i_v(G)-$sets have $\lceil\frac{2n-1}{5}\rceil$ vertices for the whole $(2n-1)-\lfloor\frac{2n-1}{5}\rfloor$ vertices remaining. Thus, we have;

$$\begin{aligned} i_{av}(G) &= \frac{1}{2n-1}.[(\lfloor\frac{2n-1}{5}\rfloor).(\lceil\frac{2n-1}{5}\rceil+1)+((2n-1)-\lfloor\frac{2n-1}{5}\rfloor).(\lceil\frac{2n-1}{5}\rceil)] \\ &= \frac{1}{2n-1}[\lfloor\frac{2n-1}{5}\rfloor.\lceil\frac{2n-1}{5}\rceil+\lfloor\frac{2n-1}{5}\rfloor+(2n-1)\lceil\frac{2n-1}{5}\rceil-\lfloor\frac{2n-1}{5}\rfloor.\lceil\frac{2n-1}{5}\rceil] \end{aligned}$$

$$= \frac{1}{2n-1}(\lfloor \frac{2n-1}{5} \rfloor + \lceil \frac{2n-1}{5} \rceil.(2n-1)).$$

Case4: ((2n-1) mod 5 ≡ 3): The cardinality of an $i_v(G)-$sets have $\lceil \frac{2n-1}{5} \rceil$, for $3.\lceil \frac{2n-1}{5} \rceil$ vertices of any graph G. Moreover, $i_v(G)-$ sets have $\lceil \frac{2n-1}{5} \rceil + 1$ vertices for the whole remaining $(2n-1) - 3.\lceil \frac{2n-1}{5} \rceil$ vertices. Then, we have;

$$i_{av}(G) = \frac{1}{2n-1}.[(3.\lceil \frac{2n-1}{5} \rceil).(\lceil \frac{2n-1}{5} \rceil) + ((2n-1) - 3.\lceil \frac{2n-1}{5} \rceil).(\lceil \frac{2n-1}{5} \rceil + 1)]$$

$$= \frac{1}{2n-1}[3.\lceil \frac{2n-1}{5} \rceil^2 + (2n-1).\lceil \frac{2n-1}{5} \rceil + (2n-1) - 3.\lceil \frac{2n-1}{5} \rceil^2 - 3\lceil \frac{2n-1}{5} \rceil]$$

$$= \frac{1}{2n-1}(\lceil \frac{2n-1}{5} \rceil.(2n-4) + (2n-1)).$$

Case5: ((2n-1) mod 5 ≡ 4): The cardinality of any $i_v(G)-$set is equal to $\lceil \frac{2n-1}{5} \rceil$ for $2.\lceil \frac{2n-1}{5} \rceil$ vertices of a graph $T(P_n)$. Furthermore, $i_v(G)-$sets have $\lceil \frac{2n-1}{5} \rceil + 1$ vertices for the whole remaining $(2n-1) - 2.\lceil \frac{2n-1}{5} \rceil$ vertices. Thus, we have;

$$i_{av}(G) = \frac{1}{2n-1}.[(2.\lceil \frac{2n-1}{5} \rceil).(\lceil \frac{2n-1}{5} \rceil) + ((2n-1) - 2.\lceil \frac{2n-1}{5} \rceil).(\lceil \frac{2n-1}{5} \rceil + 1)]$$

$$= \frac{1}{2n-1}[2.\lceil \frac{2n-1}{5} \rceil^2. + (2n-1)\lceil \frac{2n-1}{5} \rceil + (2n-1) - 2.\lceil \frac{2n-1}{5} \rceil^2 - 2\lceil \frac{2n-1}{5} \rceil]$$

$$= \frac{1}{2n-1}(\lceil \frac{2n-1}{5} \rceil.(2n-3) + (2n-1)).$$

The proof is completed. □

Theorem 3.4 *Let $T(W_{1,n})$ be the total graph of $W_{1,n}$. Then,*

$$i_{av}(G) = \frac{1}{3n+1}[(1 + \lceil \frac{n}{3} \rceil) + (n.(5 + \lceil \frac{2n-8}{5} \rceil + \lceil \frac{2n-3}{5} \rceil + \lceil \frac{n-3}{3} \rceil))].$$

Proof. The number of vertices of $W_{1,n}$ and $T(W_{1,n})$ are $|V(W_{1,n})| = n+1$ and $|V(T(W_{1,n}))| = 3n+1$, respectively. Let $T(W_{1,n})$ be G and let vertices set of G be $V(G) = V_1(G) \cup V_2(G) \cup V_3(G) \cup V_4(G)$.

$V_1(G)$: The set contains center vertex of graph $W_{1,n}$.
$V_2(G)$: The set contains all vertices of graph $W_{1,n}$ except center vertex.
$V_3(G)$: The set contains the edges of graph $W_{1,n}$, which are adjacent to center vertex; are the vertices of graph $T(W_{1,n})$.
$V_4(G)$: The set contains the edges of the cycle of graph $W_{1,n}$ are the vertices of graph $T(W_{1,n})$.

To find the value of i_{av} of graph $T(W_{1,n})$, we have 4 cases for taking the proper vertices.

Case 1.Let v be the vertex of $V_1(G)$. The vertex v is the center vertex with degree n of graph $W_{1,n}$. The proof is similar to Theorem 3.1 Case1. Then, we have $i_v(G) = 1 + \lceil \frac{n}{3} \rceil$.

Case 2. Let v be the vertex of $V_2(G)$. The degree of vertices of $V_2(G)$ are 6. Vertex v adjacent 2 vertices of $V_4(G)$, 1 vertex from $V_3(G)$, center vertex which is $V_1(G)$, 2 vertices of $V_3(G)$. There are $3n-6$ vertices remaining in graph of G. To obtain the $i_v(G)$-independence set, we do as follows. Firstly, to $i_v(G)$-independence set we should add the vertex which is at most interconnection of each other vertices in $V_3(G)$. This vertex is first vertex with maximum degree of $V_3(G)$. Then, the number of remaining vertices are $2n-8$ and now we have the graph which is showed Figure 3.

This graphs's maximum vertex degree is 4. Then, we can add at least $\lceil \frac{2n-8}{5} \rceil$ vertices to the $i_v(G)$-set. Consequently, when $v \in V_2(G)$, for n vertices, there are totally $(2 + \lceil \frac{2n-8}{5} \rceil)$ vertices in the $i_v(G)$-set. Then. we have $i_v(G) = 2 + \lceil \frac{2n-8}{5} \rceil$.

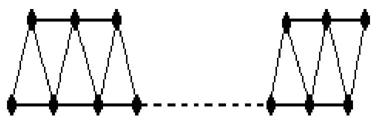

Figure 3

Case 3. Let v be the vertex of $V_3(G)$. The degree of vertices of $V_3(G)$ are $n+3$. Vertex v adjacent $n+3$ vertices. These vertices are 2 vertices of $V_4(G)$, one vertex of $V_2(G)$, center vertex and remaining $n-1$ vertices of $V_3(G)$. Therefore, there remain $2n-3$ vertices in graph G. Thus, we now have the graph $T(P_{n-1})$ remaining. At most 5 vertices in graph $T(P_n)$ can be dominated by any one vertex. Moreover, we can add at least $\lceil \frac{2n-3}{5} \rceil$ vertices to $i_v(G)$-set. Consequently, when $v \in V_3(G)$, for n vertices, there are totally $(1 + \lceil \frac{2n-3}{5} \rceil)$ vertices in the $i_v(G)$-set.

Case 4. Let v be the vertex of $V_4(G)$. The degree of vertices of $V_4(G)$ are 6 and vertex v adjacent 6 vertices. These vertices are 2 vertices of $V_3(G)$, 2 vertices of $V_2(G)$, 3 vertices of $V_4(G)$. Then, there remain $3n-6$ vertices in graph G. To obtain the $i_v(G)$-set, we do as follows. Firstly; we add center vertex of $V_1(G)$. This vertex adjacent remaining vertex of the $V_2(G)$ and $V_3(G)$. In graph G, there are finally $n-3$ remaining vertices of $V_4(G)$ and we now have the graph $T(P_{n-3})$. Then, one of these vertices can dominate at most three vertices. We should add at least $\lceil \frac{n-3}{3} \rceil$ vertices to $i_v(G)$-set. Consequently, when $v \in V_4(G)$, for n vertices there are totally $(2 + \lceil \frac{n-3}{3} \rceil)$ vertices in the $i_v(G)$-set.

By cases 1, 2, 3 and 4, we have

$i_{av}(G) = \frac{1}{|V(G)|}(\sum_{v\epsilon V_1(G)} i_v(G)+\sum_{v\epsilon V_2(G)} i_v(G)+\sum_{v\epsilon V_3(G)} i_v(G)+\sum_{v\epsilon V_4(G)} i_v(G))$
$= \frac{1}{3n+1}[(1+\lceil\frac{n}{3}\rceil) + \sum_{v\epsilon V_2(G)}^{n}(2+\lceil\frac{2n-8}{5}\rceil) + \sum_{v\epsilon V_3(G)}^{n}(1+\lceil\frac{2n-3}{5}\rceil) + \sum_{v\epsilon V_4(G)}^{n}(2+$
$\lceil\frac{n-3}{3}\rceil)]$
$= \frac{1}{3n+1}[(1+\lceil\frac{n}{3}\rceil) + (n.(2+\lceil\frac{2n-8}{5}\rceil)) + (n.(1+\lceil\frac{2n-3}{5}\rceil)) + (n.(2+\lceil\frac{n-3}{3}\rceil))]$
$= \frac{1}{3n+1}[(1+\lceil\frac{n}{3}\rceil) + (n.(5+\lceil\frac{2n-8}{5}\rceil+\lceil\frac{2n-3}{5}\rceil+\lceil\frac{n-3}{3}\rceil))]$.

Thus, the proof is completed. □

Theorem 3.5 *Let $T(K_n)$ be the total graph of K_n. Then,*

$$i_{av}(T(K_n)) = 1 + \lfloor\frac{n-1}{2}\rfloor.$$

Proof. The number of vertices of K_n and $T(K_n)$ are $|V(K_n)| = n$ and $|V(T(K_n))| = \frac{n^2+n}{2}$, respectively. Let $T(K_n)$ be G. A graph G is $(2n-2)$-regular. Let v be the any vertex of graph G. Vertex v adjacent $(2n-2)$ vertices. To obtain the $i_v(G)$-set, we can not add these $(2n-2)$ vertices. There are $\frac{(n-2)(n-1)}{2}$ vertices remaining which are not adjacent vertex v and the new graph be formed by these vertices. This new graph is $(2n-6)$- regular. When we take one vertex to new graph, there are $[\frac{(n-2)(n-1)}{2} - (2n-5)]$ vertices remaining and one new graph formed. This graph is $(2n-10)$ -regular. This procedure of selecting vertex proceeds repeatedly till the last graph is whether 0-regular that mean trivial graph or 2-regular graph K_3. This process repeat with $\lfloor\frac{2n-2}{4}\rfloor$, so $\lfloor\frac{n-1}{2}\rfloor$ for expect choosing firstly vertex v. Finally, for n vertices there are vertices in the $i_v(G)$-set. Then we have,

$$i_{av}(G) = \frac{1}{|V(G)|}\sum_{v\epsilon V(G)} i_v(G)$$
$$= \frac{1}{\frac{n^2+n}{2}}[(\frac{n^2+n}{2}).(1+\lfloor\frac{n-1}{2}\rfloor)]$$
$$= 1 + \lfloor\frac{n-1}{2}\rfloor.$$

Thus, the proof is completed. □

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